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COMPARISON OF METHODS USED IN CARTOGRAPHY FOR THE SKELETONISATION OF AREAL OBJECTS

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Key words: skeletonisation, cartography, Medial Axis, Straight Skeleton, Voronoi diagram, Delaunay triangulation, comparison of skeletonisation methods

Abstract

The article presents a method that would compare skeletonisation methods for areal objects. The skeleton of an areal object, being its linear representation, is used, among others, in cartographic visualisation. The method allows us to compare between any skeletonisation methods in terms of the deviations of distance differences between the skeleton of the object and its border from one side and the distortions of skeletonisation from another. In the article, 5 methods were compared: Voronoi diagrams, densified Voronoi diagrams, constrained Delaunay triangulation, Straight Skeleton and Medial Axis (Transform). The results of comparison were presented on the example of several areal objects. The comparison of the methods showed that in all the analysed objects the Medial Axis (Transform) gives the smallest distortion and deviation values, which allows us to recommend it.

PORÓWNANIE METOD SZKIELETYZACJI OBIEKTÓW POWIERZCHNIOWYCH STOSOWANYCH W KARTOGRAFII

Słowa kluczowe: szkieletyzacja, kartografia, medial axis, straight skeleton, diagramy Voronoia, triangulacja Delaunaya, porównanie metod szkieletyzacji

Abstrakt

W artykule przedstawiono metodę porównania metod szkieletyzacji obiektów powierzchniowych. Szkielet obiektu powierzchniowego jako jego liniowa reprezentacja wykorzystywany jest m.in. w wizualizacji kartograficznej. Metoda pozwala na porównanie dowolnych metod szkieletyzacji pod względem odchyłek różnic odległości szkieletu obiektu od jego granicy oraz zniekształceń szkieletyzacji. W pracy porównano 5 metod: diagramy Voronoia, zagęszczone diagramy Voronoia, ograniczoną triangulację Delaunaya, Straight Skeleton i Medial Axis (Transform). Wyniki porównania zaprezentowano na przykładzie kilku obiektów powierzchniowych. Porównanie metod wykazało, że we wszystkich analizowanych obiektach najmniejsze wartości zniekształceń i odchyłek posiada szkielet wyznaczony metodą Medial Axis (Transform), co pozwoliło zalecić ją do stosowania.

INTRODUCTION

The skeletonisation of areal objects is understood as the creation of one or more linear objects which are their one-dimensional representation (Li 2007). Linear objects will be called skeletons of areal objects. Generalisation, that is the reduction of spatial dimension resulting from skeletonisation, can be necessary due to visual requirements (if the areal object is unrecognisable after cartographic generalisation) or the necessity to store spatial information in the form of linear objects (e.g. a, the network of roads or rivers) (Chrobak 2012; Burghardt *et al.* 2014). Regardless of the objective, linear objects created out of areal object shall preserve
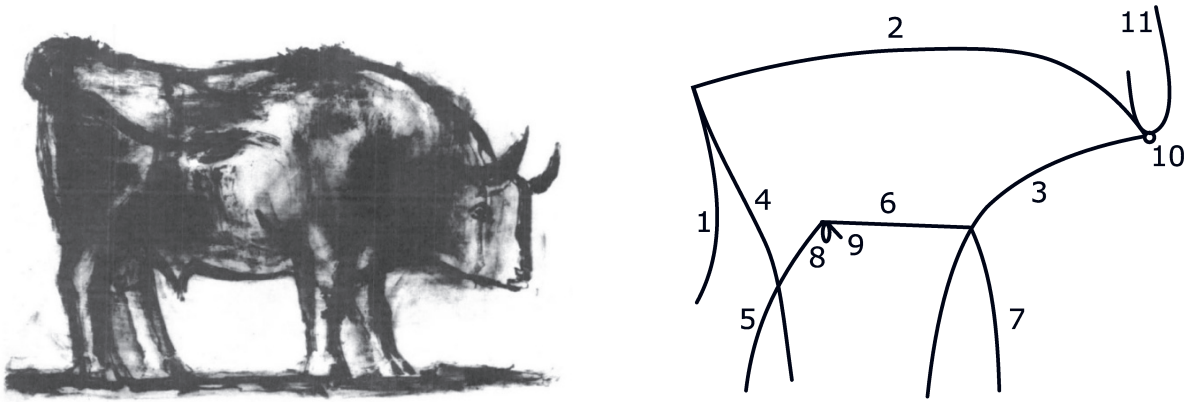
Fig. 1 – Vector representation of Pablo Picasso „Bull – plate 11" (1946, stone lithography). (own source)
Rys. 1 – Reprezentacja wektorowa „Byka – płyty 11" Pabla Picassa (1946, litografia w kamieniu). (źródło własne).

and reflect their shapes. The objective of skeletonisation is to create a linear representation of objects by their natural decomposition into parts that reflect the shape of areal objects, as perceived. This representation shall enable access to full information about parts of the object as well as the whole object, in order to support its processing (Katz *et al*. 2003). Apart from cartography supported by GI Science, there are many other research disciplines that use skeletonisation, including computer graphics with computational geometry, spatial planning, physiology, biology, remote sensing, and many others (Okabe *et al*. 2000). Attempts to show the most characteristic qualities of an object through its skeletonisation can be also found in art, with the best known example being the suite "Bull" by Pablo Picasso (Lavin 1993) which consists of 11 lithographs that show a bull. The last of them is a symbolic representation that consists of 11 lines, only (Fig. 1).

In cartography, skeletonisation is used mostly in the operators of dimension reduction (Su *et al*. 1998; Haunert and Sester 2007; Szombara 2013a, 2013b). It can also support other operators of digital cartographic generalisation (Christensen 2003; Szombara 2014) and serve as a network of cartographic points for cartographic generalisation (Jones *et al*. 1999).

A method of comparing skeletonisation methods has not yet been presented in the literature. There is only a comparison of methods understood as the determination of skeletal lines during DTM creation (Gökgöz and Gülgen 2004). Although the authors often mention and describe more than one method, their comparison is limited to verbal descriptions and some practical examples. In this paper, the author proposes the comparison method using the example of 5 skeletonisation methods.

DESCRIPTION OF SKELETONISATION METHODS

This article presents 5 skeletonisation methods, used in the process of cartographic generalisation. Some of the methods, like Voronoi diagrams and constrained Delaunay triangulation, are used as an approximation of more complicated methods, such as Medial Axis (Transform) (Christensen 1999).

Voronoi Diagram for a Set of Points

This diagram is a very well known and commonly used method of dividing a plane. It has many applications in cartography, including making skeletons of areal objects (Gold *et al*. 2008). Whenever a Voronoi diagram is mentioned in the article, without its type being specified, the Voronoi diagram of a set of points is meant. "Given a set of two or more but a finite number of distinct points in the Euclidian plane, we associate all locations in that space with the closest member(s) of the point set with respect to the Euclidean distance. The result is a tessellation off the plane into a set of regions associated with members off the point set" (Okabe *et al*. 2000).

A VD based on vertices of a boundary of an areal object is used for the creation of its skeleton by linking the internal diagram edges not intersecting with the boundary of the object (Fig. 2a). As an object's skele-
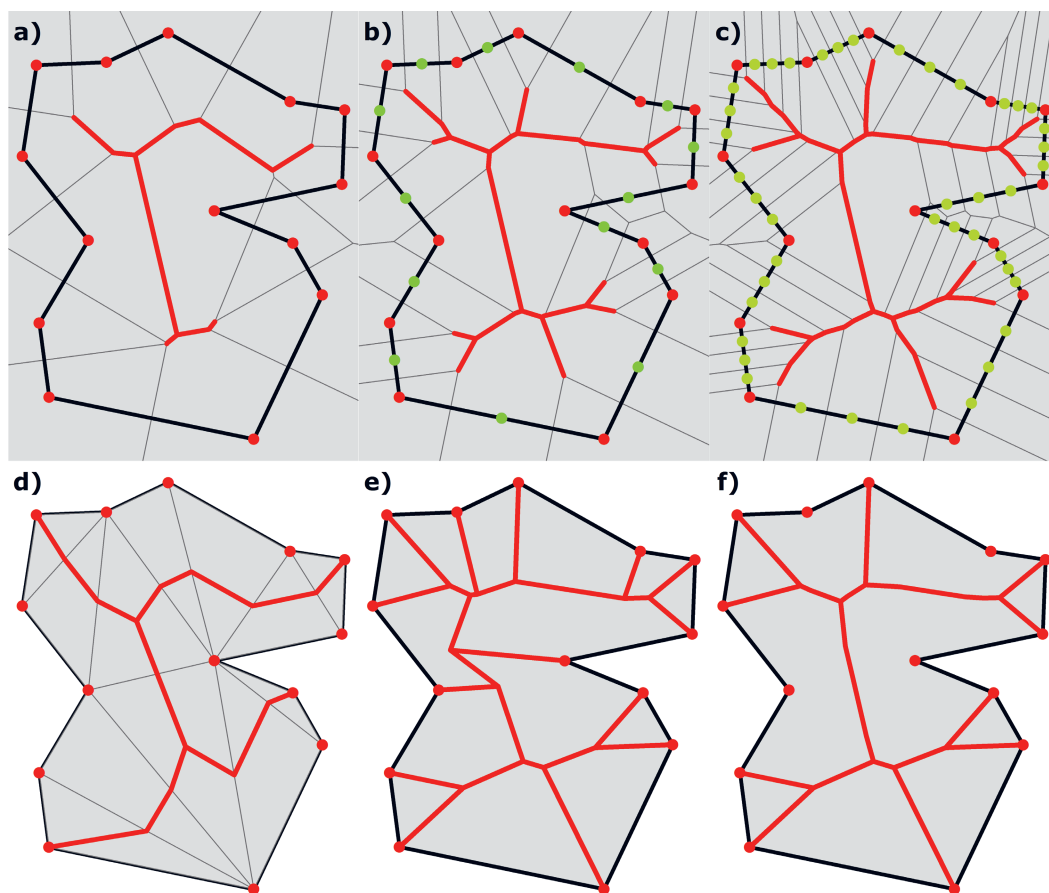
Fig. 2 – Example of a skeleton (red lines) of the object created: a) based on VD edges; b), c) based on densified Voronoi diagrams with the density of boundary vertices (red) increased twice (b) and four times (c) by pseudo nodes (green); d) for constrained Delaunay triangulation; e) with the use of Straight Skeleton method; f) with Medial Axis (Transform) method

Rys. 2 – Przykłady szkieletu (czerwone linie) obiektu powierzchniowego utworzonego: a)   na podstawie krawędzi DV, b) i c) na podstawie krawędzi zagęszczonych diagramów Voronoia przy dwu (b)) i czterokrotnym (c)) dogęszczeniu wierzchoł-ków (kolor czerwony) granicy pseudowęzłami (kolor zielony); d) dla ograniczonej triangulacji Delaunaya; e) metodą Straight Skeleton; f) metodą Medial Axis Transform

ton, determined this way, depends on the situation of the vertices, all the irregularities in their situation result in the occurrence of irregularities in the skeleton of the object. In extreme cases, irregularities in situation of the vertices can make it impossible to use the VD edges to create the skeleton, due to intersection of diagram edges with the boundary of the object (example Fig. 8b).

Densified Voronoi Diagram for a Set of Points

The strong influence of the situation of the vertices of the object's boundary on the irregularity of a skeleton's shape can be partially reduced if the density of the object's boundary is increase by adding pseudo nodes (Fig. 2b, c). This type of diagram will be called densified Voronoi diagram (DVD). The technique of creating the skeleton of the object from such a set of vertices remains the same as in the case of VD. It should be noticed that together with the increase of the pseudo-nodes number, the route of the created skeleton line becomes smoother.

Constrained Delaunay Triangulation

The commonly known Delaunay triangulation method is used in skeletonisation in such a way that the triangulation is constrained (CDT) by the boundary of an areal object (Fig. 2d). Among others, the method is
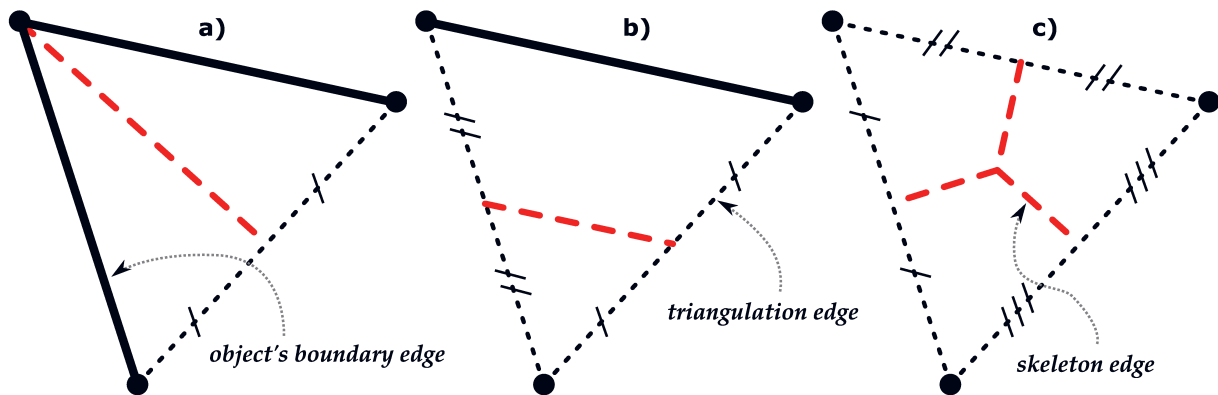
Fig. 3. Determination of a skeleton for triangles of particular types in the constrained Delaunay triangulation (De Lucia and Black 1987). a) triangle with two edges coincident with object's boundary, b) triangle with one edge, c) internal triangle
Rys. 3. Wyznaczanie szkieletu dla trójkątów poszczególnych typów w ograniczonej triangulacji Delaunaya (De Lucia and Black 1987). a) trójkąt o dwóch krawędziach pokrywających się z granicą obiektu, b) trójkąt o jednej krawędzi, c) trójkąt wewnętrzny

used in cartographic generalisation by Ordnance Survey (Jones *et al.* 1999). In every triangulation triangle, the skeleton is created in a different way, depending on the number of edges coincident with the boundary of the areal object (Fig. 3).

Straight Skeleton

The Straight Skeleton method (StSk) was developed directly for the purpose of skeletonisation of areal objects (Aichholzer *et al.* 1995). The definition of StSk skeleton is based on the hypothetic process of a polygon shrinking inwards. During shrinking, the polygon boundary edges move parallel to their original location at a constant speed, while the points in which the neighbouring edges meet are preserved. The vertices move at a speed dependant on the angle between the edges. During the process some of the moving edges meet and, as a result, new vertices are created that connect the edges. Those vertices move in the same way as the vertices of boundary edges. The edges continue to move until an overall reduction to a single vertex occurs. The paths traced out by all the moving vertices constitute the object's skeleton. The skeleton in the StSk method is a set of straight-line segments (Fig. 2e).

Medial Axis (Transform)

Medial Axis (MA), like StSk, is a method used directly for skeletonisation of areal objects (Fig. 2f). Its first definition is based on the concept of Grassfire

Transform (Blum 1967). The areal object can be compared to a hypothetical ideally uniform grass field. If we set fire to the field along its all boundaries at the same time, the walls of fire moving towards its interior will determine the MA in the form of the places where the walls of the fire will meet. MA has also another, equivalent, definition based on maximal discs (Choi *et al.* 1997).

A maximal disc is a disc inscribed in the areal object, so that it cannot be circumscribed by any other disc also inscribed in the areal object. The set of all the centres of such discs constitutes the MA and the set of paired centres and radii constitutes the Medial Axis Transform (MAT). MA can also be defined as a subset of Voronoi diagram edges for vertices and edges of an areal object's boundary. From the set of all the edges of such a diagram, the edges coming out from concave vertices are deleted. Some algorithms for determination of MA are based on this fact (Lee 1982).

For areal objects not containing concave vertices, MA and StSk are equal and for other objects they differ only locally "close to" those vertices.

**METHOD OF COMPARING
THE SKELETONISATION METHODS**

Each of the basic versions of the described skeletonisation methods provides a slightly different shape of the skeleton, number of legs, number of links between the legs and the object's boundary. It was de-
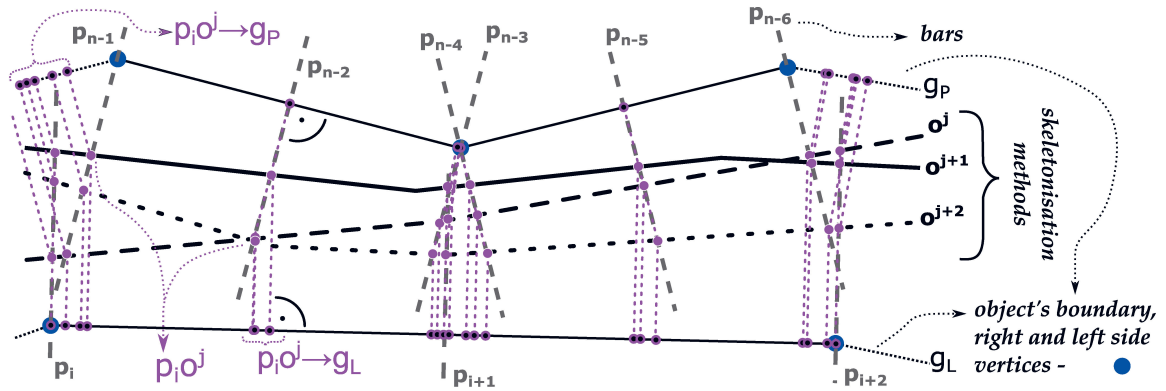
Fig. 4. Symbols introduced for the comparison of skeletons
Rys. 4. Oznaczenia wprowadzone przy porównaniu szkieletów

cided that the structure of the skeleton created with the CDT method would constitute the basis for comparison (the number of legs and places of contact point with the outer boundary edge of an object). In the skeletons MA, StSk, VD and DVD some legs were removed, and the last two were extended to the boundary of the object.

The comparison for all the skeletonisation methods was done with regard to the only constant element: the boundary of the areal object (g).[1] For each of the edges three perpendicular segments (p_i) were determined (referred to as bars) in two vertices and in the central point of the edge. The length of the bars was chosen depending on the local skeleton's route in such a way that each bar intersects each skeleton (o^j) exactly once. Each intersection point between the bar and the skeleton was referred to the object's boundary placed on its right (g_p) and left (g_L) side.

For each bar and skeleton, the points of intersection were determined (p_io^j). For each point, the distance from the right ($p_io^j \rightarrow g_P$) and left ($p_io^j \rightarrow g_L$) side of the boundary was calculated, in other words, the lengths of segments of rectangular projections p_io^j on the object's boundary were calculated. All the symbols are presented in Figure 4.

In the next step the differences in lengths between the point p_io^j and two boundaries of the areal object were calculated for each bar and each skeleton:

$$\Delta d_i^j = d(g_P)_i^j - d(g_L)_i^j \qquad (1)$$

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[1] More detailed description of the method comparing skeletonisation methods can be found in the work by (Szombara 2014).

where $d(g_P)_i^j$ is the length of a segment whose starting point p_io^j belongs to the skeleton, and its end $p_io^j \rightarrow g_P$ to the boundary of the edge placed on the right side of the skeleton. $d(g_L)_i^j$ is the length of its projection on the left edge.

On the basis of the calculated differences (1) two ways of comparing skeletonisation methods are proposed. It should be noted that for each of the differences, the value 0 is considered optimal, therefore locally the skeleton is placed just in the middle of the boundaries of the areal object.

Deviations of distance Differences of Bars

The first method of comparing skeletonisation methods consists of calculating the first type of total deviation (s^j) of the differences in distances from the boundary at the bars:

$$s^j = \sqrt{\frac{\sum_{i=1}^n \left(\Delta d_i^j\right)^2}{n-1}} \qquad (2)$$

For parts (k) of the object, the partial deviation (s_k^j) can also be calculated, and then, according to the law of error propagation, the total deviation of second type can be calculated (s^j):

$$s_k^j = \sqrt{\frac{\sum_{i=1}^n \left(\Delta d_i^j\right)^2}{n-1}}, \qquad (3),$$

$$\left(S^j\right)^2 = \sum_{k=1}^m \left(s_k^j\right)^2 \qquad (4)$$

Calculating the partial deviation might prove helpful if the object consists of several parts with the same shape.

Distortion of Skeletonisation

The second way of comparing the skeletons of areal objects is calculating the distortion of skeletonisation. Local distortion is calculated relative to the left edge $z_{i\,L}^{j}$ and the right edge $z_{i\,P}^{j}$:

$$z_{i\,L}^{j} = \frac{d(g_P)_i^j - d(g_L)_i^j}{d(g_L)_i^j + d(g_P)_i^j}, \tag{5}$$

$$z_{i\,P}^{j} = \frac{d(g_L)_i^j - d(g_P)_i^j}{d(g_L)_i^j + d(g_P)_i^j}. \tag{6}$$

Absolute values of numerators of the aforementioned equations are equal:

$$\left| d(g_L)_i^j - d(g_P)_i^j \right| = \left| d(g_P)_i^j - d(g_L)_i^j \right|. \tag{7}$$

This relation can be used in the formula for local distortion of skeletonisation relative to both edges:

$$z_i^j = \frac{\left| d(g_L)_i^j - d(g_P)_i^j \right|}{d(g_L)_i^j + d(g_P)_i^j} = \frac{\left| d(g_P)_i^j - d(g_L)_i^j \right|}{d(g_L)_i^j + d(g_P)_i^j}. \tag{8}$$

For the purpose of describing the whole skeleton j, regarding the distortions, the sum of local distortions SZ^j and mean skeletonisation distortion Z^j is calculated:

$$SZ^j = \sum_{i=1}^{n} z_i^j, \tag{9}$$

$$Z^j = \frac{SZ^j}{n}. \tag{10}$$

It should be noted that local distortion of skeletonisation and mean distortion of skeletonisation take values in the range $\langle 0; 1 \rangle$, where 0 means an ideal skeletonisation and 1 means the lack of skeletonisation (in fact, the skeleton coincides with one of the edges). This characteristic was used for comparison of skeletonisation results for different objects.

COMPARISON OF SKELETONISATION METHODS APPLYING THE AUTHOR'S OWN METHOD

The developed method of comparing skeletonisation methods was applied to Ordnance Survey data. The product Boundary-Line represents the highest sea and ocean level over a scale of many yearsand is provided in ESRI Shapefile format; this format was used for all spatial data. The license (OS OpenData Licence 2013) allows research and non-commercial use provided that the "Ordnance Survey data © Crown copyright and database right 2013" is mentioned. 10 small islands of differentiated shape were chosen to perform the cal-

Fig. 5. The objects chosen for calculations and their situation referred to Great Britain and surrounding islands
Rys. 5. Wybrane do obliczeń obiekty i ich lokalizacja na tle Wielkiej Brytanii i otaczających wysp

culations (Fig. 5). All the spatial data analyses were conducted in ArcGIS ArcMap 10.2 software. StSk was drawn in Rhinoceros 4 software.

For the areas of 10 objects, length and maximum distance between any two vertices were calculated. These three parameters were used in defining the measures that would characterise the shape of the object: the ratio of the area to the perimeter and the ratio of the perimeter to the maximum distance between any two vertices. All the parameters and measures are presented in columns 2–6 of Table 1.

On the basis of the described measures a cluster analysis in Statistica 10 was performed. This analysis was done on the basis of the following three factors: k-means algorithm, Euclidean distance, and maximising the distance between clusters as a method of determining the initial cluster centres, 3 clusters were obtained with an error in the training sample equal 0.1693. Allocating the objects to the clusters is shown in column 7 of Table 1. Diagrams of means for quantitative variables are presented in Figure 6. One object from each cluster was chosen for further analysis. These were objects 4, 5 and 8.

Tab. 1 – Shape measures and classification based on the example objects
Tab. 1 – Miary kształtu i klasyfikacja wykonana na ich podstawie dla przykładowych obiektów

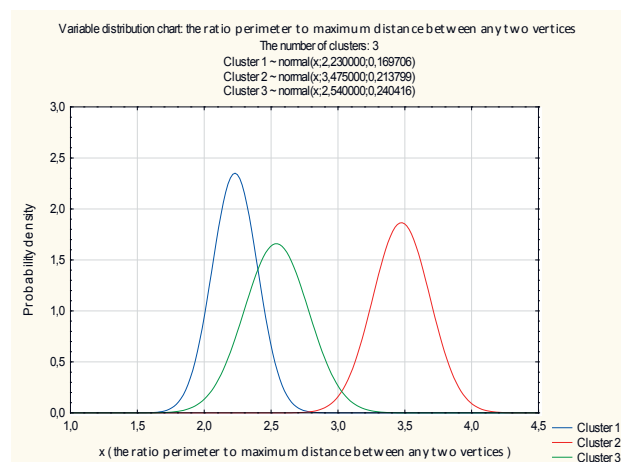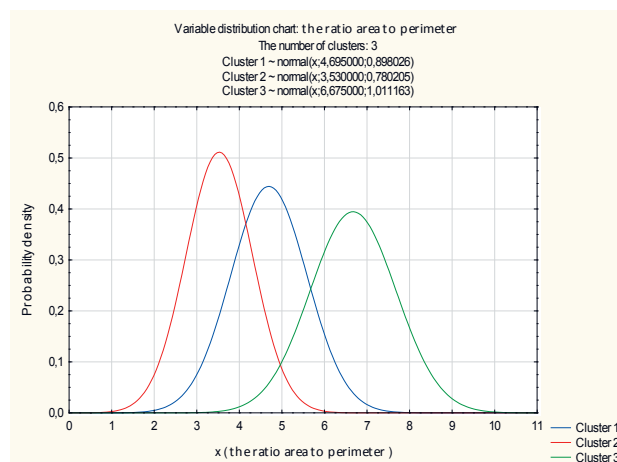| Object	Area [m²]	Perimeter [m]	Maximal length between any two vertices [m]	[2] / [3] [m]	[3] / [4] [–]	Clusters
[1]	[2]	[3]	[4]	[6]	[5]	[7]
1	498.71	93.55	39.86	5.33	2.35	1
2	249.39	100.36	29.86	2.48	3.36	3
3	564.25	94.64	39.92	5.96	2.37	2
4	460.89	101.43	30.87	4.54	3.29	3
5	753.37	101.94	37.65	7.39	2.71	2
6	403.42	104.40	28.76	3.86	3.63	3
7	438.66	107.45	30.99	4.08	3.47	3
8	437.50	107.63	51.01	4.06	2.11	1
9	418.94	123.76	37.75	3.39	3.28	3
10	393.23	138.89	36.36	2.83	3.82	3

Fig. 6. Diagrams for quantitative variables of cluster analysis
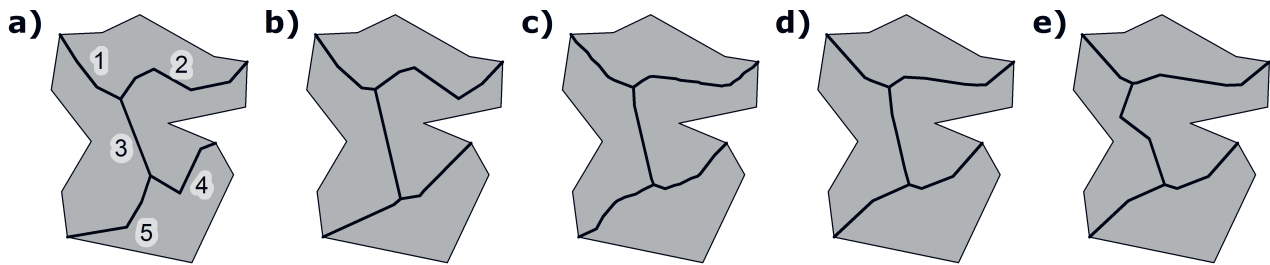**Rys. 6.** Wykresy średnich zmiennych ilościowych analizy skupień

Fig. 7. Skeleton of object 4 created with various methods: a) CDT, b) VD, c) DVD, d) MA, e) StSk. Part numbers, marked for skeleton created with CDT method, correspond with other skeletons

Rys. 7. Szkielet obiektu 4 wyznaczony metodami: a) OTD, b) DV, c) ZDV, d) MA, e) StSk. Numery części zaznaczone dla szkieletu wyznaczonego metodą OTD dla pozostałych szkieletów są im odpowiadające

Tab. 2. Results of skeletonisation assessment for object 4
Tab. 2. Wyniki oceny szkieletyzacji dla obiektu 4

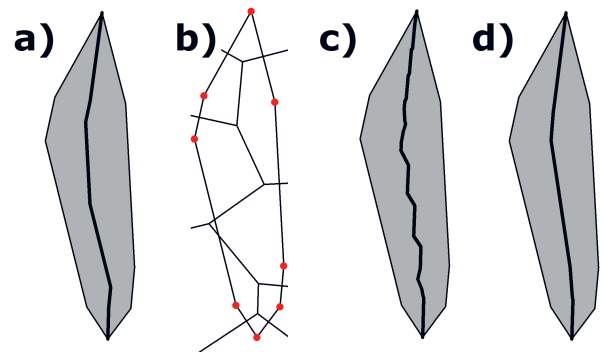| Method	CDT	VD	DVD	MA	StSk
s^j	1.552	1.476	0.077	0.042	1.117
s_1^j	1.637	0.959	0.058	0.034	0
s_2^j	0.901	1.698	0.063	0.040	0.355
s_3^j	1.132	0.120	0.023	0.091	2.897
s_4^j	1.356	0.851	0.070	0	0
s_5^j	3.119	3.401	0.209	0	0
S^j	4.042	4.014	0.238	0.105	2.919
SZ^j	3.368	3.426	0.196	0.041	0.982
Z^j	0.116	0.118	0.007	0.001	0.034

Fig. 8 – Skeleton of object 8 created with methods: a) CDT, b) Voronoi diagrams for object – internal edges intersect with object's boundaries, which makes it impossible to create the skeleton, c) DVD, d) MA (StSk).

Rys. 8 – Szkielet obiektu 8 wyznaczony metodami: a) OTD, b) diagram Voronoia dla obiektu – krawędzie wewnętrzne przecinają się z granicą obiektu, co uniemożliwia wyznaczenie szkieletu, c) ZDV, d) MA (StSk).

Tab. 3. Results of skeletonisation assessment for objects 8 and 5
Tab. 3. Wyniki oceny szkieletyzacji dla obiektów 8 i 5

Method	Object 8			Object 5			
	CDT	DVD	MA(StSk)	CDT	VD	DVD	MA(StSk)
s^j	0.686	0.376	0	1.298	0.733	0.046	0
SZ^j	0.966	0.502	0	1.507	0.815	0.049	0
Z^j	0.048	0.025	0	0.054	0.029	0.002	0

Both objects 8 and 5 are convex so StSk and MA skeletons are both equal. For object 5, skeletonisation using the VD method was impossible without a sig- nificant change of the primal skeleton boundary. The other results are presented in tables 2 and 3, and figures 7, 8 and 9.
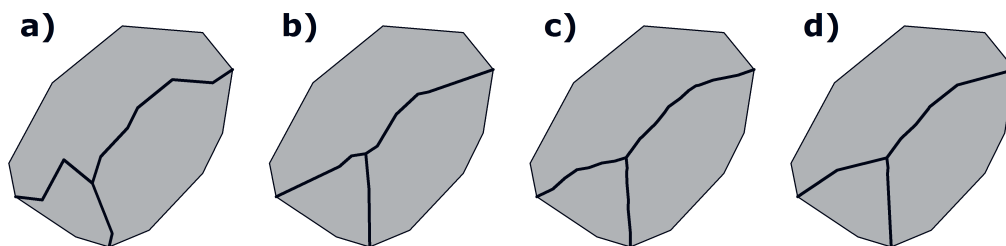
Fig. 9. Skeleton of object 5 created with methods: a) CDT, b) VD, c) DVD, d) MA (StSk)
Rys. 9. Szkielet obiektu 5 wyznaczony metodami: a) OTD, b) DV, c) ZDV, d) MA (StSk)

RESULTS AND CONCLUSIONS

Skeletons presented in the article were manually edited by the removal of dispensable edges and interpolating edges, connecting the skeleton with the object's boundary. It should be mentioned that there are automatic solutions for some of these problems (Penninga *et al.* 2005; Haunert and Sester 2007), the article focuses on comparison of skeletonisation methods and not on fully automatic implementation. The selected sample objects had a small number of vertices which enabled a more accurate analysis of every part of the object. Tests run for larger objects confirm the results for the smaller objects, but due to the high number of bars, the differences between similar methods were less visible.

The analysis of the results of comparing skeletonisation methods for three sample objects led to the following conclusions.

Cluster analysis facilitated differentiation into 3 classes: The first class includes objects of lengthy, fusiform shape; the second class includes objects of rounded, oval shape; the third class includes objects of complicated shape with many "curves".

If a spatial file of a skeleton created by the MA(T) method could contain parabolic sections, the skeleton itself would have a deviation and distortion of 0. Due to necessary changes of parabola into polygons for object 4 the values differ from 0.

For object 4 with complicated shape, the VD and CDT methods enable the creation of skeletons with highest values of distortion and deviation which are similar for both methods in one object.

For object 5, the VD method allows the creation of a skeleton with lower values of distortion and deviation for the object. The ratio of partial deviation to the skeletons created with the VD and CDT methods is reciprocal to part 2 of object 4. This fact results from the high local irregularity of the skeleton, influenced by irregular placement of edges for object's boundaries.

A skeleton created with the StSk method for convex objects is equal to a skeleton created with the MA method. For local convex parts or with slight partial distortion they are the same or similar to values for skeletons created with the MA method. Significant differences are observed only where objects are locally concave. An example is part 3 of object 4, where the skeleton for the StSk method is significantly different to the skeleton for the MA method.

Skeletons created with the DVD method have higher values of distortion and deviation than skeletons created with the VD and CDT methods, and lower than those created with the MA and StSk methods, with exception of the aforementioned part 3 of object 4. Here, the last two skeletons have partial distortion values higher than the skeleton for the DVD method. For MA this is a result of necessary change of the parabolic section into polygons.

For all analysed objects, the lowest distortion and deviation values were definitely observed with skeletons created with the MA method.

To conclude, for objects or their parts with uncomplicated shapes the best would be the usage of skeletons created with the MA, StSk or DVD methods, especially if the skeleton is to be used for cartographic presentation. It must be mentioned that for objects with a significant number of concave vertices influencing its shape, the DVD method is more recommended than the StSk method. If a skeleton shall not be used for cartographic presentation on a map, the usage of the CDT and VD methods is possible. The first is especially suitable for objects with lengthy shapes, the latter for objects of rounded shapes. It should be emphasised that the shape

of a skeleton created with the VD method for irregular placement of boundary vertices may cause high local distortions of skeletisation. If possible for a given usage, the optimal selection of the MA method for skeleton creation is recommended, especially as it may be extended to a MAT structure, which could be used in many operators of cartographic generalisation (Szombara 2014). MA is a demanding method due to its calculations and other methods presented in this article can be used when MA cannot be applied. Such an approach can be found in the literature (Christensen 1999).

The application of a method for comparing the skeletonisation methods is justified for outlining the administrative boundaries along rivers, where in some cases automatic algorithms for skeletonisation are used instead of traditional (Bieda and Hycner 2012).

LITERATURE

Aichholzer, O., Aurenhammer, F., Alberts, D., Gärtner, B., 1995. A Novel Type of Skeleton for Polygons. *Journal of Universal Computer Science*, 1 (12), pp. 752–761.

Bieda, A., Hycner, R., 2012. Administrative legal borders run along rivers. *Geomatics and Environmental Engineering*, 6 (2), p. 15.

Blum, H., 1967. A Transformation for Extracting New Descriptors of Shape. [In:] W. Whaten-Dunn, ed. *Models for the Perception of Speech and Visual Form*. MIT Press, Cambridge, Mass., pp. 362–380.

Burghardt, D., Duchêne, C., Machaness, W., 2014. *Abstracting Geographic Information in a Data Rich World*. Springer.

Choi, H.I., Choi, S.W., Moon, H.P., 1997. Mathematical theory of medial axis transform. *Pacific Journal of Mathematics*, 181 (1), pp. 57–88.

Christensen, A.H.J., 1999. Cartographic Line Generalization with Waterlines and Medial-Axes. *Cartography and Geographic Information Science*, 26 (1), pp. 19–32.

Christensen, A.H.J., 2003. Two experiments on stream network generalization. [In:] *Proceedings of the 21st International Cartographic Conference*. Durban, South Africa, pp. 10–16.

Chrobak, T., 2012. The map and geoinformatics. *Geoinformatica Polonica*, 11, pp. 7–16.

Gökgöz, T., Gülgen, F., 2004. Comparison of Two Methods for Deriving Skeleton Lines of Terrain. *The International Archives of the Photogrammetry, Remote Sensing and Spatial Information Sciences*, 35 (1991), pp. 618–622.

Gold, C.M., Mioc, D., Anton, F., Sharma, O., Dakowicz, M., 2008. A Methodology for Automated Cartographic Data Input, Drawing and Editing Using Kinetic Delaunay/Voronoi Diagrams. [In:] M. Gavrilova, ed. *Generalized Voronoi Diagram: A Geometry-Based Approach to Computational Intelligence SE - 7*. Berlin Heidelberg: Springer, pp. 159–196.

Haunert, J.-H., Sester, M., 2007. Area Collapse and Road Centerlines based on Straight Skeletons. *GeoInformatica*, 12 (2), pp. 169–191.

Jones, C.B., Bundy, G.L., Ware, J.M., 1999. Map Generalization with a Triangulated Data Structure. *Cartography and Geographic Information Science*, 22 (4), pp. 317–331.

Katz, R.A., Pizer, S.M., Carolina, N., 2003. Untangling the Blum Medial Axis Transform. *International Journal of Computer Vision*, 55 (2/3), pp. 139–153.

Lavin, I., 1993. Picasso's Bull(s): Art History in Reverse. *Art in America*, LXXXI, pp. 76–93.

Lee, D.T., 1982. Medial Axis Transformation of a Planar Shape. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, PAMI-4 (4), pp. 363–369.

Li, Z., 2007. *Algorithmic Foundation of Multi-Scale Spatial Representation*. London: CRC Press.

De Lucia, A., Black, T., 1987. A comprehensive approach to automatic feature generalization. [In:] *Proceedings of the 13th International Cartographic Conference*. Morelia, Mexico, pp. 169–192.

Okabe, A., Boots, B., Sugihara, K., Chiu, S.N., 2000. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*. ed. 2. Production. Chichester, England: Wiley.

OS OpenData Licence [online], 2013. Protokół dostępu: https://www.ordnancesurvey.co.uk/business-and-government/licensing/using-creating-data-with-os-products/os-opendata.html.

Penninga, F., Verbree, E., Quak, W., Oosterom, P., 2005. Construction of the Planar Partition Postal Code Map Based on Cadastral Registration. *GeoInformatica*, 9 (2), pp. 181–204.

Su, B., Li, Z., Lodwick, G., 1998. Morphological Models for the Collapse of Area Features in Digital Map Generalization. *GeoInformatica*, 2 (4), pp. 359–383.

Szombara, S., 2013a. Unambiguous Collapse Operator of Digital Cartographic Generalisation. [In:] *Proceedings of 16th ICA Workshop on Generalisation and Map Production*. Drezno, 23–24 sierpnia: International Cartographic Association.

Szombara, S., 2013b. Transformation of areal objects into linear objects, regarding the map scale. *Geoinformatica Polonica*, 12, pp. 23–34.

Szombara, S., 2014. Uogólniony operator harmonizacji w cyfrowej generalizacji kartograficznej. Akademia Górniczo-Hutnicza im. Stanisława Staszica.