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QUATERNIONS – ALTERNATIVE FOR EULER ANGLES IN KINEMATIC ANALYSIS OF MACHINE MECHANISMS

KWATERNIONY – ALTERNATYWA DLA KĄTÓW EULERA W ANALIZIE KINEMATYCZNEJ MECHNIZMÓW MASZYN

Abstract

The article presents some types of problems that can occur during solving the inverse kinematics and the steps to prevent them. These problems concern the numerical singularities which occur when Euler angles are used to describe the orientation of the mechanism links in space. This paper proposes to eliminate the mentioned numerical singularities by replacing Euler angles with quaternions. The discussed issues were shown on the example of a backhoe excavator equipment system.

Keywords: *quaternions, Euler angles, inverse kinematics*

Streszczenie

W artykule przedstawiono pewien typ problemów, jakie mogą wystąpić podczas rozwiązywania zadania odwrotnego kinematyki oraz kroki pozwalające na ich zapobieganie. Problemy te dotyczą osobliwości numerycznych powstałych w wyniku stosowania kątów Eulera do opisu orientacji członów mechanizmu w przestrzeni. Poniższa praca proponuje wyeliminowanie wspomnianych osobliwości numerycznych poprzez zastąpienie katów Eulera kwaternionami. Poruszane zagadnienia przedstawiono na przykładzie osprzętu roboczego koparki podsiębiernej.

Słowa kluczowe: *kwaterniony, kąty Eulera, zadanie odwrotne kinematyki*

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1. Introduction

The main task of the kinematic analysis of machines mechanisms is the study of motion of these mechanisms without taking into account the forces and moments that act upon them. In the case of serial mechanisms, an analysis can be divided into two general tasks. The first one is the simple kinematics problem, which is used to obtain the position and orientation of the last element in the kinematic chain. The second one, which is the inverse kinematics problem, is used to determine the values of all joint variables when the position and orientation of the last element of the kinematic chain is known. In serial mechanisms the inverse kinematics problem is much more difficult to solve than the simple kinematics problem [1]. This is due to the fact that the equations describing the relationships between joint variables and the position and orientation of the links are highly nonlinear [2].

The article discusses some types of problems that can occur during solving the inverse kinematics problem and also the steps to prevent them. These problems concern the numerical singularities which occur when Euler angles are used to describe the orientation of the mechanism links in space. This paper proposes to eliminate the mentioned numerical singularities by replacing Euler angles with quaternions. The discussed issues were shown on the example of a backhoe excavator equipment system.

2. Kinematic analysis based on the multibody system formalism

Kinematic analysis, which is a part of the theory of mechanisms and machines, is the subject of a lot of research work and publications [1–7]. As a result of these studies several procedures have been developed that allow to automatically perform the kinematic analysis. One such procedure is the multibody system method. A solution of the inverse kinematics problem in the multibody system method is equivalent to solving the following system of equations:

$$\Phi(q,t) = \begin{bmatrix} \Phi^{K}(q) \\ \Phi^{D}(q,t) \end{bmatrix} = 0 \tag{1}$$

$$\dot{\Phi}(\dot{q},q,t) = \Phi_{q}\dot{q} = -\Phi_{t} = \begin{bmatrix} 0 \\ -\Phi_{t}^{D} \end{bmatrix} \tag{2}$$

$$\ddot{\Phi}(\ddot{q},\dot{q},q,t) = -\left(\Phi_{q}\dot{q}\right)_{q}\dot{q} - 2\Phi_{qt}\dot{q} - \Phi_{tt} = \begin{bmatrix} -\left(\Phi_{q}^{K}\dot{q}\right)_{q}\dot{q} \\ -\left(\Phi_{q}^{D}\dot{q}\right)_{q}\dot{q} - 2\Phi_{qt}^{D}\dot{q} - \Phi_{tt}^{D} \end{bmatrix} \tag{3}$$

The formula identified by number (1) is a non-linear system of algebraic equations whose solution is the vector of generalized coordinates. The formulas identified by number (2) and (3) are a linear systems of algebraic equations whose solution is the vector of generalized velocities and generalized accelerations. A detailed description of the above equations is

given in reference [3]. The vector of generalized coordinates describing the position and orientation of all mechanism links can be expressed as follows:

$$q = \left[q_1^T, q_2^T, q_3^T, q_4^T, q_5^T, q_6^T, q_7^T, q_8^T, q_9^T, q_{10}^T, q_{11}^T, q_{12}^T, q_{13}^T, q_{14}^T \right]^T \tag{4}$$

The presented above vector (4) describes the position and orientation of the links of the backhoe excavator equipment system, which is shown in Fig. 1. The position and orientation of a single link is determined by the vector and take the following form:

$$q_i = [r_i^T, \alpha_i, \beta_i, \gamma_i]^T \tag{5}$$

where:

r_i^T — specifies the position of the origin of the local reference system π_i of the link in global reference system π_0,

$\alpha_i, \beta_i, \gamma_i$ — three Euler angles α, β, γ which describe orientation of the local reference system π_i of the link relative to global reference system π_0.
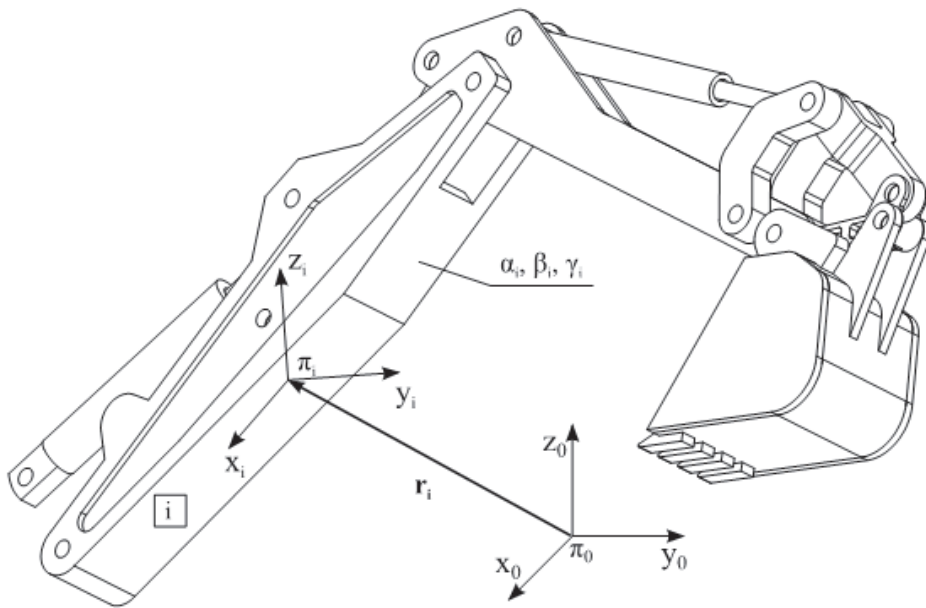

Fig. 1. Backhoe excavator equipment system scheme

The most time consuming step in the kinematic analysis is the calculation of generalized coordinates' vector (4). For this purpose an iterative Newton-Raphson algorithm is most widely used. It allows to determine the approximated values of the roots satisfying the system of equations (1). The Newton-Raphson algorithm was graphically shown in Fig. 2. Vector

function $\boldsymbol{\Phi}$ (which is the left side of the equation system (1)) could be expanded in Taylor series in the vicinity of point $\boldsymbol{q}^i$ with an assumption that the norm of vector $\Delta\boldsymbol{q}_i$ becomes small and the higher-order terms in vector $\Delta\boldsymbol{q}$ can be neglected:

$$\boldsymbol{\Phi}\left(\boldsymbol{q}^i+\Delta\boldsymbol{q}\right)\approx\boldsymbol{\Phi}\left(\boldsymbol{q}^i\right)+\frac{\partial\boldsymbol{\Phi}\left(\boldsymbol{q}^i\right)}{\partial q_1}\Delta q_1+\cdots+\frac{\partial\boldsymbol{\Phi}\left(\boldsymbol{q}^i\right)}{\partial q_N}\Delta q_N=\boldsymbol{\Phi}\left(\boldsymbol{q}^i\right)+\boldsymbol{\Phi}_q\left(\boldsymbol{q}^i\right)\Delta\boldsymbol{q} \qquad (6)$$

Assuming that function $\boldsymbol{\Phi}$ must satisfy the system of equations (1) at point $\boldsymbol{q}^{i+1}$ ($\boldsymbol{q}^i$ is an initial approximation), equation (6) can be compared to zero, under the condition that $\Delta\boldsymbol{q}=\boldsymbol{q}^{i+1}-\boldsymbol{q}^i$. Then the equation (6) takes the following form:

$$\boldsymbol{\Phi}\left(\boldsymbol{q}^i\right)=\boldsymbol{\Phi}\left(\boldsymbol{q}^i\right)+\boldsymbol{\Phi}_q\left(\boldsymbol{q}^i\right)\left(\boldsymbol{q}^{i+1}-\boldsymbol{q}^i\right)=0 \qquad (7)$$

Which can also be written as:

$$\boldsymbol{q}^{i+1}=\boldsymbol{q}^i+\boldsymbol{\Phi}_q^{-1}\left(\boldsymbol{q}^i\right)\boldsymbol{\Phi}\left(\boldsymbol{q}^i\right)=0 \qquad (8)$$
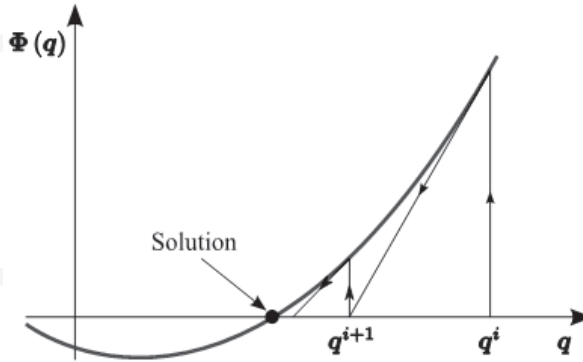

Fig. 2. Graphic presentation of the Newton-Raphson algorithm (only two first steps were shown)

The Newton-Raphson algorithm presented in Fig. 2 is an iterative algorithm, in which a single iteration consists of the following steps:
– Calculation of approximation $\boldsymbol{q}^{i+1}$ from the formula (8).
– Checking whether the approximation $\boldsymbol{q}^{i+1}$ satisfies the system of equations (1) with an accuracy of ε, i.e. if the condition $\|\boldsymbol{\Phi}(\boldsymbol{q}^{i+1})\|\leq\varepsilon$ is true.
– If the algorithm in the previous step did not get a solution at a defined number of iterations, then the emergency termination of solution process occurs [4–5].

3. Euler angles singularities

In the classical approach to the kinematic analysis of spatial mechanisms, orientation of the particular mechanism links is described by Euler angles. This description, however, has a disadvantage, which is important from the kinematic analysis' point of view [3]. When considering kinematic analysis of the boom separated from the equipment system shown in Fig. 1 boom orientation in space is described by Euler angles in the convention *z-x-z*. The global reference system π_0 and local reference system π_1, which is connected with boom was defined as shown in Fig. 3. For boom kinematic analysis purposes the following assumptions was made:

– boom is connected to a global reference system π_0 with rotational joint in such a way, that it can only perform a rotational movement about an axis z_0,

– vector of generalized coordinates of the boom at initialization time has the following coordinates $q_1 = [0, 1.78, 0, 0, 0, \pi/2]^T$,

– boom relative rotation angle is described by the following relation $\varphi_1 - t^2 - \dfrac{\pi}{2}$,

– analysis is focused only on obtaining a vector of generalized coordinates of the boom in a defined time span.
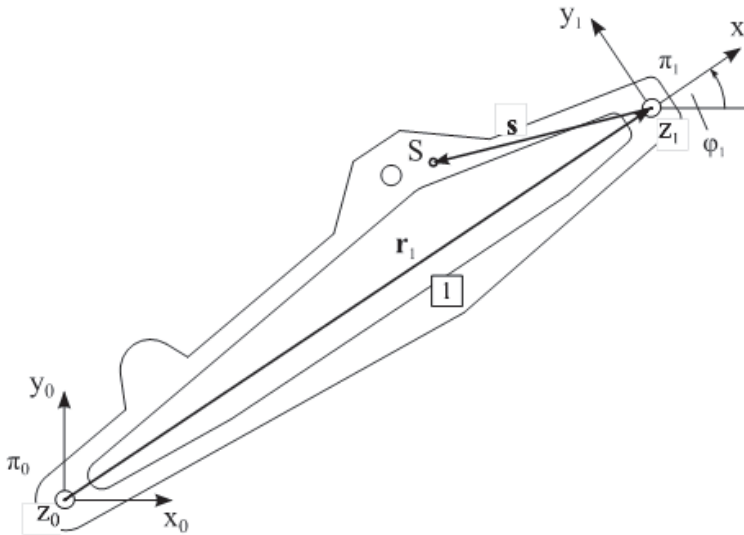

Fig. 3. Boom extracted for kinematic analysis purposes

For calculating the vector of generalized coordinates of a boom the Newton-Raphson algorithm was used. At the first iteration, for the time $t = 0$ Jacobian matrix is as follows:

$$\boldsymbol{\varPhi}_q\left(\boldsymbol{q}^1\right) = \begin{bmatrix} -1 & 0 & 0 & -1.78 & 0 & -1.78 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1.78 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix} \tag{9}$$

Unfortunately, this is a singular matrix because its determinant is equal to 0, and therefore, it cannot be reversed. As a result, an attempt to solve the system of equations (1) finishes with a failure.

The appearance of singularities in the Jacobian matrix is caused by unfavorable adoption of orientation of the reference systems π_0 and π_1 (in practice it is often required that the z axes of the local reference systems of links have to be parallel to the z axis of the global reference system). It is, therefore, apparent that the system of Euler angles in z-x-z convention is not clearly specified in a situation where the axes of the global and the local reference system are parallel. In this position, the xy planes of the local and the global reference system are parallel, which means that the positioning of the line of nodes is not clearly defined and, therefore, the precession angle α is also not uniquely defined. For the same reason it is impossible to uniquely specify intrinsic rotation angle γ [3].

4. Quaternions

One of the solutions of the Jacobian matrix degeneration problem is to replace Euler angles with quaternions. Quaternions are an algebraic structure that extends complex numbers. Quaternions were defined by Hamilton in 1853 in the following form [8]:

$$q_{\boxminus} = s + ia + jb + kc \quad s, a, b, c \in \mathbb{R} \tag{10}$$

where:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{11}$$

$$ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad kj = -i, \quad ik = -j \tag{12}$$

From the kinematic analysis' point of view their algebraic form is more useful, which can be written as follows:

$$q_{\boxminus} = [s, v] \quad s \in \mathbb{R}, v \in \mathbb{R}^3 \tag{13}$$

$$q_{\boxminus} = [s, x\boldsymbol{i}, y\boldsymbol{j}, z\boldsymbol{k}] \quad s, x, y, z \in \mathbb{R} \tag{14}$$

Elements of a quaternion which describe rotation of angle θ of the mechanism link around vector $\boldsymbol{a}$ can be expressed as follows:

$$q_{\mathbb{H}} = \begin{bmatrix} \cos\left(\dfrac{\theta}{2}\right) \\[2mm] \sin\left(\dfrac{\theta}{2}\right)\cdot a_x \\[2mm] \sin\left(\dfrac{\theta}{2}\right)\cdot a_y \\[2mm] \sin\left(\dfrac{\theta}{2}\right)\cdot a_z \end{bmatrix} \tag{15}$$

As we know, changing the orientation of the mechanism link is the same as changing of orientation of any point which is connected to the link. Thus, if for boom shown in Fig. 3 we determine point S, whose position in reference system π_1 is defined by vector $\boldsymbol{s}$, then describing orientation of vector $\boldsymbol{s}$ will be inconsistent with describing the orientation of a boom. For example, rotation of angle φ_1 of boom from Fig. 3 around vector $\boldsymbol{a}_0$, which was defined in global reference system π_0, could be written as:

$$\boldsymbol{s}' = q_{\mathbb{H},1}\,\boldsymbol{s}\,q_{\mathbb{H},1}^{-1} \tag{16}$$

where:

$$q_{\mathbb{H}} = \begin{bmatrix} \cos\left(\dfrac{\varphi_1}{2}\right) \\[2mm] \sin\left(\dfrac{\varphi_1}{2}\right)\cdot \boldsymbol{a}_0 \end{bmatrix}, \quad \boldsymbol{a}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{17}$$

Like shown previously the kinematic analysis of boom shown in Fig. 3 requires determining Jacobian Matrix for time $t = 0$, but orientation of boom will be described with quaternions. Elements of the Jacobian matrix after changing the description of boom orientation will have the following values:

$$\Phi_q\left(\boldsymbol{q}^1\right) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1.78 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1.78 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} \tag{19}$$

Values of the above matrix determinant equal 1, therefore it is not a singular matrix and, as a consequence, the Newton-Raphson converges. The use of quaternions in this example

allowed to avoid the singularities from the Jacobian matrix and also to determine a generalized coordinates vector in each time step.

5. Conclusions

The use of the Euler angles to describe the orientation of the mechanism links in space, can give rise to singularities in the Jacobian matrix. It is often caused by the fact that the requirements for orientation of links' local reference systems in the global reference system are in conflict with the requirements which have to be fulfilled when Euler angles are used. The use of quaternions for orientation description eliminates the risk that the mentioned singularities will appear and thus increases numerical stability of the Newton-Raphson algorithm. For better consistency of inverse kinematics the algorithm it is good to encapsulate translations and rotations of mechanism links into a unified representation. This can be achieved by using dual quaternions and Clifford algebra to solve the inverse kinematics problem.

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