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SOME REMARKS ON THE GENERALIZED BANACH CONTRACTION PRINCIPLE

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A b s t r a c t

This paper presents some results concerning the Generalized Banach Contraction Principle: *In a complete metric space X if for some* $N \geq 1$ *and* $0 < M < 1$ *the mapping* $T : X \to X$ *satisfies* $\min\{d(T^j x, T^j y),\ 1 \leq j \leq N\} \leq Md(x, y)$ *for any* $x, y \in X$, *then T has a unique fixed point*. In some special cases, the above constant M can be replaced by a continuous, non--increasing function $0 \leq \phi\,(d(x, y)) \leq 1$ such that $\phi(t) = 1$ if, and only if, t = 0.

Keywords: *generalized Banach contraction principle*, *gbcp*, *fixed point*, *metric fixed point theory*, *syndetic set*

S t r e s z c z e n i e

W artykule przedstawiono pewne wyniki związane z Uogólnionym Twierdzeniem Banacha o Punkcie Stałym: *W przestrzeni metrycznej zupełnej X jeśli dla pewnych* $N \geq 1$ *i* $0 < M < 1$ *odwzorowanie* $T : X \to X$ *spełnia warunek* $\min\{d(T^j x, T^j y),\ 1 \leq j \leq N\} \leq Md(x, y)$ *dla dowolnych* $x, y \in X$, *to T ma dokładnie jeden punkt stały.* W pewnych szczególnych przypadkach zastąpiono stałą M ciągłą, nierosnącą funkcją $0 \leq \phi\,(d(x, y)) \leq 1$ dla której $\phi(t) = 1$ wtedy i tylko wtedy, gdy t = 0.

Słowa kluczowe: *uogólnione twierdzenie Banacha o punkcie stałym*, *gbcp*, *punkt stały*, *metryczna teoria punktu stałego*, *zbiór syndetyczny*

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1. Introduction

This paper is related to the Generalized Banach Contraction Principle (we will refer to it as GBCP for short) formulated as follows:

Theorem 1.1. [1] *In a complete metric space X if for some* $N \geq 1$ *and* $0 < M < 1$ *the mapping* $T : X \to X$ *satisfies* $\min\{d(T^j x, T^j y),\ 1 \leq j \leq N\} \leq Md(x, y)$ *for any* $x, y \in X$, *then T has a unique fixed point.*

Firstly, let us have a quick look at a history of this theorem.

For $N = 1$ the above theorem is obviously the celebrated classical Banach contraction principle, see [B].

In [2], it was shown that if X is compact and if for $x, y \in X$ and $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $d(T^n x, T^n y) \leq \varepsilon$, then T has a unique fixed point.

In [7], it was shown that GBCP is true for $N = 2$ and for $N = 3$, if T is continuous.

In [6], it was shown that it is true for any $N \geq 1$, if T is uniformly continuous.

In [11], it was shown that it is true for any $N \geq 1$, if T is strongly continuous.

In [8], it was shown that it is true for $N = 3$ and for any $N \geq 1$, if T is continuous.

In [9] and [1] the above theorem was finally proved.

Before we proceed, let us recall some definitions:

Finite set $A = \{n_1 < n_2 < \ldots < n_k\} \subset \mathbb{N}$ *is said to be S-syndetic with constant* $S \in \mathbb{N}$, *if* $n_{i+1} - n_i \leq S$ *for* $1 \leq i \leq k - 1$.

Infinite set $A \subset \mathbb{N}$ *is said to be S-syndetic with constant* $S \in \mathbb{N}$, *if for any* $l \in \mathbb{N}$, *there is* $\{i \in \mathbb{N} \ :\ l + 1 \leq i \leq l + S\} := [l + 1, l + S] \cap A \neq \varnothing$.

Infinite set $A \subset \mathbb{N}$ *is said to be piecewise S-syndetic with constant* $S \in \mathbb{N}$, *if for any* $N \in \mathbb{N}$, *there exists* $B \subset A$ *such that* $\# B \geq N$ *and B is S-syndetic.*

We call set A syndetic (or piecewise syndetic), if it is S-syndetic (piecewise S-syndetic) for some $S \in \mathbb{N}$.

Both proofs of Theorem 1.1 presented in [9], [1] are not elementary. In [1], the proof uses, among other tools, the strong result of H. Fürstenberg stated that *if* $\mathbb{N} \supset B = \cup_{i=1}^{N} B_i$ *is piecewise syndetic, then* B_i *is piecewise syndetic for some* $i \in \{1, \ldots, N\}$. In [9], the proof uses Ramsey's theorem.

Another approach to the metric fixed point theory, presented in ([10]), resulted in the following:

Theorem 1.2. [10] *Let* (X, d) *be a complete metric space. Let* $\phi : [0, \infty) \to [0, 1]$ *be a non-increasing function such that* $\phi(t) = 1$ *if, and only if,* $t = 0$. *Assume that* $T : X \to X$ *is a contractive mapping such that:*

1. $d(x_0, Tx_0) \leq \dfrac{d(x, x_0) - d(Tx, Tx_0)}{2}, \quad x \notin M$

2. $d(Tx,Ty) \le \phi(d(x,y))d(x,y), \quad x,y \in M$

for some $M \subset X$, $x_0 \in M$. *Then, T has a unique fixed point.*

This instantly led to the following result regarding contractive mappings in complete metric spaces:

Corollary 1.3. *For* φ *as in Theorem 1.2, if* $d(Tx,Ty) \le \phi(d(x,y))d(x,y)$, $x,y \in X$, *then T has the unique fixed point.*

2. Main results

In this paper, we try to mix two approaches presented in Theorem 1.1 and Corollary 1.3, in order to achieve, under some additional assumptions, the following:

Theorem 2.1. *Let* (X, d) *be a complete metric space,* $N \geq 1$. *Let* $\phi:[0,\infty) \to [0,1]$ *be a continuous, non-increasing function satisfying* $\phi(t)=1$ *if, and only if,* $t = 0$. *Assume that* $T: X \to X$ *satisfies* $\min\{d(T^j x, T^j y),\ 1 \le j \le N\} \le \phi(d(x,y)) \cdot d(x,y)$, $x,y \in X$. *Then T has a unique fixed point.*

The above theorem, in this general form, was proved in the author's PhD thesis [12]. The proof is mainly based on the above mentioned Fürstenberg theorem. In this paper, we will show some special but important cases in which it is not necessary to apply this strong tool, which significantly simplifies the proof.

Firstly, we assume that T is a continuous mapping having the N-syndetic Cauchy orbit.

Theorem 2.2. *Let* (X, d) *be a complete metric space,* $N \geq 1$. *Let* $T: X \to X$ *be a continuous mapping such that:*

1. for all $y \neq z \in X$, *there exists* $n \in \mathbb{N}_1$ *such that* $d(T^n y, T^n z) < d(y,z)$

2. there exists $x \in X$, $N \in \mathbb{N}_1$ *and* $\{n_j\}_{j=1}^{\infty} \subset \mathbb{N}$ *-increasing, N-syndetic such that* $\{T^{n_j} x\}_{j=1}^{\infty}$ *is a Cauchy sequence.*

T then has exactly one fixed point.

Proof. Let $x_0 := \lim_{n\to\infty} T^{n_j} x$. Since $1 \le n_{j+1} - n_j \le N$, there exists $b \in \{1,\dots,N\}$, $\{n_{j_l}\} \subset \{n_j\}$ such that $n_{j_l+1} - n_{j_l} = b$. On the other hand, $T^{j_l+1} x = T^b(T^{n_{j_l}} x)$. By definition of x_0, there is $T^{j_l+1} x \to x_0$, $l \to \infty$, and due to the continuity of T, we have $T^b(T^{n_{j_l}} x) \to T^b(x_0)$, $l \to \infty$. Therefore, T^b has a fixed point x_0.

Let $c := \min\{b \in \mathbb{N}_1 : T^b \text{ has a fixed point}\}$. Let us assume that $c < 1$.

Let $r := \min\{d(T^j x_0, T^k x_0),\ 0 \le j < k \le c-1\}$. Of course, $r > 0$ – the opposite case would contradict with the choice of c. Since the minimum was taken over a finite set, there exists $j_0, k_0 \in \{0,\dots,c-1\}$ such that $j_0 < k_0$ and $d(T^{j_0} x_0, T^{k_0} x_0) = r$.

By assumption, we can choose $n \in \mathbb{N}$ such that $d(T^{n+j_0} x_0, T^{n+k_0} x_0) < d(T^{j_0} x_0, T^{k_0} x_0)$. But $n + j_0 = p \cdot c + \tilde{j}_0$. On the other hand, $n + k_0 = q \cdot c + \tilde{k}_0$ for some $\tilde{j}_0, \tilde{k}_0 \in \{0,\dots,c-1\}$. Of course, $\tilde{j}_0 \ne \tilde{k}_0$. Moreover, $T^{q \cdot c} x_0 = T^c(\dots(T^c x_0)\dots) = x_0 = T^c(\dots(T^c x_0)\dots) = T^{p \cdot c} x_0$. Therefore, $d(T^{\tilde{j}_0} x_0, T^{\tilde{k}_0} x_0) = d(T^{p \cdot c + \tilde{j}_0} x_0, T^{q \cdot c + \tilde{k}_0} x_0) = d(T^{n+j_0} x_0, T^{n+k_0} x_0) < d(T^{j_0} x_0, T^{k_0} x_0)$, which leads to a contradiction with the choice of j_0, k_0.

We will now show that there is exactly one fixed point. Let us assume on the contrary, that there are $x_0 \ne y_0$ such that $Tx_0 = x_0$, $Ty_0 = y_0$. By our assumption, there exists $n \in \mathbb{N}$ such that $d(x_0, y_0) > d(T^n x_0, T^n y_0) = d(x_0, y_0)$, which leads to a contradiction.

□

If there exists $x_0 \in X$ such that $d(T^n x_0, T^{n+1} x_0) \to 0$ with $n \to \infty$, we can directly point out the Cauchy orbit. This fact is a subject of the following:

Theorem 2.3. *Let* (X, d) *be a complete metric space. Assume that* $\phi : \mathbb{R}_+ \to [0,1]$ *is a continuous, non-increasing function such that* $\phi(t) = 1$ *if, and only if,* $t = 0$. *Let* $T : X \to X$ *be a mapping satisfying the following conditions:*

1. *there exists* $N \in \mathbb{N}_1$ *such that* $\min\{d(T^j x, T^j y),\ 1 \le j \le N\} \le \phi(d(x,y))\, d(x,y)$ *for all* $x, y \in X$,
2. $d(T^{n+1} x, T^n x) \to 0$, $n \to \infty$ *for some* $x \in X$.

Then $\{T^n x\}_{n=1}^{\infty}$ *is a Cauchy sequence.*

Proof. Let $x_n := T^n x$. Assume for indirect proof, that $\{T^n x\}_{n=1}^{\infty} = \{x_n\}_{n=1}^{\infty}$ is not a Cauchy sequence. Therefore, there exists $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, there exists $k_n, l_n > n$ such that $d(x_{k_n}, x_{l_n}) \ge \varepsilon$. For $n \in \mathbb{N}$ let $k_n > n$ be the smallest possible such that there exist $l_n > k_n$ satisfying $d(x_{k_n}, x_{l_n}) \ge \varepsilon$. For $k_n > n$, let l_n be the smallest possible such that $d(x_{k_n}, x_{l_n}) \ge \varepsilon$. Fix $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $d(T^{n+1} x, T^n x) = d(x_{n+1}, x_n) \le \frac{\varepsilon}{2}$. Let us notice, that for $n \ge n_0$ there is $l_n \ge k_n + 2$. Then, for fixed $l_n \ge k_n + 2$ we have $d(x_{k_n}, x_{l_n}) \le d(x_{k_n}, x_{l_n - 1}) + d(x_{l_n - 1}, x_{l_n}) < \varepsilon + d(x_{l_n - 1}, x_{l_n}) \to \varepsilon$ with $n \to \infty$. Therefore, $\limsup_{n \to \infty} d(x_{k_n}, x_{l_n}) \le \varepsilon$. However, since $d(x_{k_n}, x_{l_n}) \ge \varepsilon$, we have $d(x_{k_n}, x_{l_n}) \to \varepsilon$, $n \to \infty$.

On the other hand, for all $n \in \mathbb{N}$ there exists $j_n \in \{1,\dots,N\}$ such that $d(x_{k_n+j_n}, x_{l_n+j_n}) \leq$ $\leq \phi(d(x_{k_n}, x_{l_n}))\, d(x_{k_n}, x_{l_n})$. Define $c_n^i := i \cdot \chi_{\{0,\dots,j_n-1\}}(i) \in \{0,\dots,N-1\}$, where χ is an indicator function. Let us estimate:

$$d(x_{k_n}, x_{l_n}) \leq d(x_{k_n}, x_{k_n+1}) + d(x_{k_n+1}, x_{l_n+1}) + d(x_{l_n+1}, x_{l_n}).$$

Repeating this procedure, we get:

$$\begin{aligned} d(x_{k_n}, x_{l_n}) &\leq \sum_{i=1}^{j_n-1} d(x_{k_n+i}, x_{k_n+i+1}) + d(x_{k_n+j_n}, x_{l_n+j_n}) + \sum_{i=1}^{j_n-1} d(x_{l_n+i+1}, x_{l_n+i}) \\ &= \sum_{i=1}^{N-1} d(x_{k_n+c_n^i}, x_{k_n+c_n^i+1}) + d(x_{k_n+j_n}, x_{l_n+j_n}) + \sum_{i=1}^{N-1} d(x_{l_n+c_n^i+1}, x_{l_n+c_n^i}) \\ &\leq \sum_{i=1}^{N-1} d(x_{k_n+c_n^i}, x_{k_n+c_n^i+1}) + \phi(d(x_{k_n}, x_{l_n}))\, d(x_{k_n}, x_{l_n}) + \sum_{i=1}^{N-1} d(x_{l_n+c_n^i+1}, x_{l_n+c_n^i}). \end{aligned}$$

By assumptions $d(x_{k_n+c_n^i}, x_{k_n+c_n^i+1}) \to 0$ and $d(x_{l_n+c_n^i}, x_{l_n+c_n^i+1}) \to 0$ with $n \to \infty$ for $i \in \{0,\dots,N-1\}$. Therefore, with $n \to \infty$ in the above estimates, due to the continuity of ϕ, we have $\varepsilon \leq 0 + \phi(\varepsilon)\cdot\varepsilon + 0$. This implies $\phi(\varepsilon) \geq 1$, which gives $\varepsilon = 0$ and we therefore get a contradiction.

$\square$

Theorem 2.3 ensures the existence and uniqueness of a fixed point.

Theorem 2.4. *Mapping T, which fulfils the assumptions of Theorem 2.3, has exactly one fixed point.*

Proof. Using notations from Theorem 2.3, let $z_0 := \lim_{n\to\infty} T^n x_0$. Due to the properties of T for any $n \in \mathbb{N}$, there exists $j_n \in \{1,\dots,N\}$ such that $d(T^{n+j_n}x_0, T^{j_n}z_0) \leq$ $\leq \phi(d(T^n x_0, z_0))\, d(T^n x_0, z_0)$. Therefore, there exists $j \in \{1,\dots,N\}$ and $\{n_k\} \subset \mathbb{N}$ such that $d(T^{n_k+j}x_0, T^j z_0) \leq \phi(d(T^{n_k}x_0, z_0))\, d(T^{n_k}x_0, z_0)$ for $k \in \mathbb{N}$. Due to this fact, $T^{n_k+j}x_0 \to T^j z_0$ with $k \to \infty$. But on the other hand, $T^{n_k+j}x_0 \to z_0$, so $T^j z_0 = z_0$.

Let $Z := \{T^n x_0\}_{n=1}^{\infty} \cup \{z_0\}$. Let us notice, that $T^j(Z) \subset Z$.

We will show that z_0 is the only one fixed point of $T^j\big|_Z$. Indeed, if there is $z_0 \neq w_0 \in Z$ – another fixed point of T^j, then $w_0 = T^n x_0$ for some $n \in \mathbb{N}$. Then

$$w_0 = T^j w_0 = \ldots = T^{k\cdot j} w_0 = T^{k\cdot j}(T^n x_0) = T^{k\cdot j+n} x_0 \to z_0,\ k \to \infty,$$

which leads to a contradiction.

We will show that z_0 is a fixed point of T. Let j be the smallest natural number such that $T^j z_0 = z_0$. Let $0 < \varepsilon := \min\{d(T^k z_0, T^l z_0),\ 0 \leq k < l \leq j-1\} = d(T^{k_0} z_0, T^{l_0} z_0)$ for some $0 \le k_0 < l_0 \le j-1$. There exists $m \in \{1,\dots,N\}$ such that $\varepsilon \le d(T^{k_0+m}z_0, T^{l_0+m}z_0) \le$ $\le \phi(d(T^{k_0}z_0, T^{l_0}z_0))\, d(T^{k_0}z_0, T^{l_0}z_0) = \phi(\varepsilon)\varepsilon$. Due to this fact, ε = 0 – this leads to a contradiction.

We will now show that z_0 is the only fixed point of *T*. Indeed, if there is $z_0 \ne w_0 \in X$ – another fixed point of *T*, then for any $n \in \mathbb{N}$ there is $T^n z_0 = z_0$ and $T^n w_0 = w_0$. For points z_0, w_0, there exists $j \in \{1,\dots,N\}$ such that $d(z_0, w_0) = d(T^j z_0, T^j w_0) \le$ $\le \phi(d(z_0, w_0)) \cdot d(z_0, w_0)$. This inequality implies that either $d(z_0, w_0) = 0$ or $\phi(d(z_0, w_0))$. Due to the properties of ϕ, in both cases $d(z_0, w_0) = 0$ – this contradicts with $z_0 \ne w_0$.

□

The theorem below is useful for proving Theorem 2.1 for $N = 2$.

Theorem 2.5. *Let* (*X*, *d*) *be a complete metric space,* N ≥ 1 *and let* $\phi : [0,\infty) \to [0,1]$ *be a continuous, non-increasing function such that* $\phi(t) = 1$ *if, and only if,* $t = 0$. *Let* $T : X \to X$ *satisfy* $\min\{d(T^j x, T^j y),\ 1 \le j \le N\} \le \phi(d(x,y)) \cdot d(x,y)$ *for any* $x, y \in X$.

Moreover, for fixed $x, y \in X$ *let the sequence* $\{j_n\}_{n=1}^{\infty} \subset \{1,\dots,N\}$ *be chosen (by the above assumption) in such a way, that for* $n \ge 2$ *we have* $d(T^{j_1+\dots+j_n}x, T^{j_1+\dots+j_n}y) \le \phi(d(T^{j_1+\dots+j_{n-1}}x, T^{j_1+\dots+j_{n-1}}y)) \cdot d(T^{j_1+\dots+j_{n-1}}x, T^{j_1+\dots+j_{n-1}}y)$. *Let* $z_n := d(T^{j_1+\dots+j_n}x, T^{j_1+\dots+j_n}y)$. *Then* $z_n \to 0$ *with* $n \to \infty$.

Proof. Obviously $z_n \ge 0$. Since $z_n \le \phi(z_{n-1})z_{n-1}$ and $\phi(t) \le 1$, $\{z_n\}_{n=1}^{\infty}$, is non--increasing.

If there exists $n \in \mathbb{N}$ such that $z_n = z_{n+1}$, then $z_n = z_{n+1} \le \phi(z_n)z_n$; therefore, either $z_n = 0$ or $\phi(z_n) \ge 1$. Due to the properties of ϕ, in both cases $z_n = 0$.

In the opposite case (when for every $n \in \mathbb{N}$ there is $z_n \ne z_{n+1}$) the sequence $\{z_n\}_{n=1}^{\infty}$ is decreasing and bounded from below; therefore, it has a limit. Let $g := \lim_{n\to\infty} z_n$. We will show that this limit is 0. Assume for the purpose of contradiction that $g > 0$. Since $\phi(g) < 1$ and $\phi(z_n) \to \phi(g)$, due to continuity of ϕ, for any $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that for any $n \ge n_0$ $\phi(z_n) \le \phi(g) + \delta$. Choose δ such that $\phi(g) + \delta < 1$. Then $\phi(z_n) \le \phi(g) + \delta < 1$ for $n \ge n_0$. On the other hand, $0 \le z_n \le \phi(z_{n-1})z_{n-1} \le$ $\le \phi(z_{n-1})\dots\phi(z_{n_0})z_{n_0} \le [\phi(g)+\delta]^{n-n_0} z_{n_0} \to 0$ with $n \to \infty$; therefore, $z_n \to 0$ with $n \to \infty$.

□

Now we will show proof of Theorem 2.1 for $N = 2$ without applying the Fürstenberg theorem.

Theorem 2.6. *Let* (X, d) *be a complete metric space. Let* $\phi : \mathbb{R}_+ \to [0,1]$ *be a continuous, non-increasing function such that* $\phi(t) = 1$ *if, and only if,* $t = 0$. *Let* $T : X \to X$ *be a mapping such that* $\min\{d(T^j x, T^j y),\ 1 \le j \le 2\} \le \phi(d(x,y))\, d(x,y)$ *for any* $x, y \in X$.

Then $d(T^{n+1}x_0, T^n x_0) \to 0,\ n \to \infty$ *for any* $x_0 \in X$ *and* T *has exactly one fixed point.*

Proof. Assume for indirect proof, that there exists $x_0 \in X$ such that $d(T^{n+1}x_0, T^n x_0) \nrightarrow 0$; therefore, there are $x_0 \in X,\ \varepsilon > 0,\ \{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ such that $d(T^{k_n}x_0, T^{k_n+1}x_0) \ge \varepsilon.$

For $k = 1, 2$ let us create two sequences $\{n_m^k\}_{m=1}^{\infty}$ in the following way:

Let $n_0^k := 0$; Having already chosen $n_0^k, \ldots, n_m^k$, define $z_m^k := d(T^{n_m^k}x_0, T^{n_m^k+k}x_0)$ and

$$n_{m+1}^k := \begin{cases} n_m^k + 1, & \text{gdy} \quad d(T^{n_m^k+1}x_0, T^{n_m^k+1+k}x_0) \le \phi(z_m^k) \cdot z_m^k \\ n_m^k + 2, & \text{gdy} \quad d(T^{n_m^k+2}x_0, T^{n_m^k+2+k}x_0) \le \phi(z_m^k) \cdot z_m^k \end{cases}$$

The sequences $\{n_m\}_{m=0}^{\infty},\ k = 1, 2$ are increasing and 2-syndetic. Moreover, due to Theorem 2.5, we have $z_m^k \to 0,\ m \to \infty,\ k \in \{1, 2\}$.

Let us notice, that for all $m_0 \in \mathbb{N}$, there exists $m \ge m_0$ satisfying $k_m \notin \{n_p^1\}_{p=0}^{\infty}$. Otherwise, there exists $m_0 \in \mathbb{N}$ such that for any $m \ge m_0$, there is $k_m \in \{n_p^1\}_{p=0}^{\infty}$; therefore, $k_m = n_{p(m)}^1$. Taking, if needed, the subsequence of $\{k_m\}_{m=0}^{\infty}$ and renumbering it, without loss of generality, we can assume that $m \le p(m)$. Then $d(T^{k_m}x_0, T^{k_m+1}x_0) =$ $= d(T^{n_{p(m)}^1}x_0, T^{n_{p(m)}^1+1}x_0) \to 0,\ m \to \infty,$ which leads to contradiction.

Therefore, for all $m_0 \in \mathbb{N}$, there exists $m \ge m_0$ satisfying $k_m \notin \{n_p^1\}_{p=0}^{\infty}$. For such m, there exists $p(m) \in \mathbb{N}$ satisfying $n_{p(m)}^1 < k_m < n_{p(m)+1}^1$. Without the loss of generality, we can assume that $m \le p(m)$. Obviously, $n_{p(m)}^1 + 1 = k_m = n_{p(m)+1}^1 - 1$.

Since $\{n_q^2\}_{q=0}^{\infty}$ is a 2-syndetic sequence, there exists $q(m) \in \mathbb{N}$ such that either $k_m = n_{q(m)}^2$ or $k_m - 1 = n_{q(m)}^2$, because $\{k_m - 1, k_m\} \cap \{n_q^2\}_{q=0}^{\infty} \ne \varnothing$. Without the loss of generality, we can assume that $m \le q(m)$.

If $k_m - 1 = n^2_{q(m)}$, then $d(T^{k_m}x_0, T^{k_m+1}x_0) \le d(T^{k_m}x_0, T^{k_m-1}x_0) + d(T^{k_m-1}x_0, T^{k_m+1}x_0) \le$ $\le d(T^{n^1_{p(m)}}x_0, T^{n^1_{p(m)}+1}x_0) + d(T^{n^2_{q(m)}}x_0, T^{n^2_{q(m)}+2}x_0) \to 0$, with $m \to \infty$ – this follows from Theorem 2.5, and leads to contradiction.

If $k_m = n^2_{q(m)}$, then $d(T^{k_m}x_0, T^{k_m+1}x_0) \le d(T^{k_m}x_0, T^{k_m+2}x_0) + d(T^{k_m+2}x_0, T^{k_m+1}x_0) \le$ $\le d(T^{n^2_{q(m)}}x_0, T^{n^2_{q(m)}+2}x_0) + d(T^{n^1_{p(m)+1}}x_0, T^{n^1_{p(m)+1}+1}x_0) \to 0$, with $m \to \infty$ – this also leads to contradiction.

Therefore, $d(T^{n+1}x_0, T^n x_0) \to 0$ with $n \to \infty$. Theorem 2.4 completes the proof.

□

We will now show a proof of Theorem 2.1 for uniformly continuous mappings.

Theorem 2.7. *Let* (X, d) *be a metric space. Assume that* $\phi: \mathbb{R}_+ \to [0,1]$ *is a continuous, non-increasing function such that* $\phi(t) = 1$ *if and only if* $t = 0$. *Let* $T: X \to X$ *be a uniformly continuous mapping such that*

$$\forall x, y \in X \quad \min\{d(T^j x, T^j y),\ 1 \le j \le N\} \le \phi(d(x,y))\, d(x,y) \tag{1}$$

for some $N \ge 1$.

Moreover, if X is complete, then T has exactly one fixed point.

Proof. First we show, that $d(T^{n+1}x_0, T^n x_0) \to 0$, $n \to \infty$ *for any* $x_0 \in X$. Assume on the contrary, that $d(T^n x_0,, T^{n+1}x_0) \nrightarrow 0$ for some $x_0 \in X$; therefore, there exists $x_0 \in X$, $\varepsilon > 0$, $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that:

$$d(T^{n_k}x_0, T^{n_k+1}x_0) > \varepsilon,\ k \in \mathbb{N} \tag{2}$$

Due to the uniform continuity of *T*, for the ε chosen above, there exists $\delta > 0$ such that $d(Tx, Ty) < \varepsilon$ for any $x, y \in X$ satisfying $d(x, y) < \delta$.

Repeating this procedure *N* times, δ can be chosen in such way, that:

$$d(x, y) < \delta \ \text{ implies } \ d(T^j x, T^j y) < \varepsilon \ \text{ for any } \ 1 \le j \le N,\ x, y \in X \tag{3}$$

Due to the assumption (1) for $x_0 \in X$ chosen in (2), there exists $j_1 \in \{1, \dots N\}$ such that $d(T^{j_1}x_0, T^{j_1+1}x_0) \le \phi(d(x_0, Tx_0))\, d(x_0, Tx_0)$. There then exists $j_2 \in \{1, \dots, N\}$ such that $d(T^{j_1+j_2}x_0, T^{j_1+j_2+1}x_0) \le \phi(d(T^{j_1}x_0, T^{j_1+1}x_0))\, d(T^{j_1}x_0, T^{j_1+1}x_0)$.

Continuing the above procedure, we will get $\{j_1 + ... + j_l\}_{l=1}^{\infty}$ – an increasing and *N*-syndetic sequence, which fulfils the assumptions of Theorem 2.5 for the pair (x_0, Tx_0). Therefore, $d(T^{j_1+...+j_l}x_0, T^{j_1+...+j_l+1}x_0) \to 0$ with $l \to \infty$. Define $m_l := j_1 + \dots + j_l$, $l \in \mathbb{N}$.

Let $\widehat{l_0} \in \mathbb{N}$ be such that $d(T^{m_l}x_0, T^{m_l+1}x_0) < \delta$ for $l \geq \widehat{l_0}$. Since the sequence $\{m_l\}_{l=\widehat{l_0}}^{\infty}$ is N-syndetic (because $\{m_l\}_{l=1}^{\infty}$ is an increasing); therefore, we can choose $l_0 \geq \widehat{l_0}$ such that $\{m_{l_0}+1, ..., m_{l_0}+N\} \cap \{n_k\}_{k=1}^{\infty} \neq \varnothing$.

Therefore, there exists $j_0 \in \{1, \ldots, N\}$ and $k_0 \in \mathbb{N}$ such that:

$$m_{l_0} + j_0 = n_{k_0} \tag{4}$$

By (2), it follows that $d(T^{n_{k_0}}x_0, T^{n_{k_0}+1}x_0) > \varepsilon$, and also $d(T^{m_{l_0}}x_0, T^{m_{l_0}+1}x_0) < \delta$, it therefore (3) implies that $d(T^{m_{l_0}+j_0}x_0, T^{m_{l_0}+j_0+1}x_0) < \varepsilon$, which, in respect of (4), leads to contradiction.

□

Now we can make use of the above theorems in the following *example*:

Example 2.1. Let $T : l_\infty \cap \{x \in l_\infty : x_i \geq 0, \ i \in \mathbb{N}\} \to l_\infty \cap \{x \in l_\infty : x_i \geq 0, \ i \in \mathbb{N}\}$ be defined in the following way: $T : x = (x_1, x_2, \ldots) \to Tx := \left(1, \frac{2x_3}{1+x_3}, \frac{\frac{1}{2}x_2}{1+x_2}, \frac{2x_5}{1+x_5}, \frac{\frac{1}{2}x_4}{1+x_4}, \ldots\right)$.

Obviously, $\|Tx - Ty\|_\infty \leq 2\|x-y\|_\infty$. We also have $T^2x = \left(1, \frac{x_2}{1+\frac{3}{2}x_2}, \frac{x_3}{1+3x_3}, \frac{x_4}{1+\frac{3}{2}x_4}, \frac{x_5}{1+3x_5}, \ldots\right)$ Let us estimate:

$$\left|\frac{x_i}{1+\frac{3}{2}x_i} - \frac{y_i}{1+\frac{3}{2}y_i}\right| \leq \left|\frac{x_i - y_i}{1+\frac{3}{2}(x_i+y_i)+\frac{9}{4}x_i y_i}\right| \leq \frac{|x_i - y_i|}{1+|x_i+y_i|} \leq \frac{|x_i - y_i|}{1+|x_i-y_i|} \leq \frac{\|x-y\|_\infty}{1+\|x-y\|_\infty},$$

$i \in \mathbb{N}$.

The latter inequality follows from the fact that $t \to \frac{t}{1+t}$ is an increasing function on $[0, \infty)$. Similarly, $\left|\frac{x_i}{1+3x_i} - \frac{y_i}{1+3y_i}\right| \leq \frac{\|x-y\|_\infty}{1+\|x-y\|_\infty}$, and therefore, $\left\|T^2x - T^2y\right\|_\infty \leq$

$\leq \dfrac{1}{1+\|x-y\|_\infty}\|x-y\|_\infty$. The mapping T satisfies the assumptions of both Theorem 2.6 and Theorem 2.7 with $\phi(t) := \dfrac{1}{1+t}$, it therefore follows that T has a unique fixed point. It is of course $(1, 0, 0, \ldots)$. However, the mapping T does not satisfy assumptions of Theorem 1.1.

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