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Ahmet HAMAL and Mehmet TERZILER

PERITOPOLOGICAL SPACES AND BISIMULATIONS

A b s t r a c t. Generalizing ordinary topological and pretopological spaces, we introduce the notion of *peritopology* where neighborhoods of a point need not contain that point, and some points might even have an empty neighborhood. We briefly describe various intrinsic aspects of this notion. Applied to modal logic, it gives rise to *peritopological models*, a generalization of topological models, a spacial case of neighborhood semantics. A new cladding for bisimulation is presented. The concept of Alexandroff peritopology is used in order to determine the logic of all peritopological spaces, and we prove that the minimal logic **K** is strongly complete with respect to the class of all peritopological spaces. We also show that the classes of T_0, T_1 and T_2-peritopological spaces are not modal definable, and that **D** is the logic of all proper peritopological spaces. Finally, among our conclusions, we show that the question whether T_0, T_1 peritopological spaces are modal definable in $\mathcal{H}(@)$ remains open.

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1. Introduction

H. CARTAN introduced filters and ultrafilters in 1937. Before that time, it was common practice to consider a topological space as a structure with an idempotent "closure operation". Right after, (as illustrated by BOURBAKI), a correlative definition for topological spaces bloomed. A topological space was viewed as a set X with a filter $\mathcal{V}(x)$ of neighborhoods attached to each of its points $x \in X$, a *field of filters*, so to say. Two conditions must hold for the new definition to be equivalent to the earlier: Each point x must belong to each of its neighborhoods $V \in \mathcal{V}(x)$, (reflexivity), and each neighborhood $V \in \mathcal{V}(x)$ must contain a neighborhood $U \in \mathcal{V}(x)$ which, in turn, is a neighborhood of each of its points $y \in U$, (transitivity). Thus, topological spaces have two different, and still equivalent, faces: Idempotent closure operations and filter fields, one and the same object.

G. CHOQUET, in 1947, dispensing with transitivity, introduced **pretopological spaces** whose associated closure operations are no longer idempotent, of course. Pretopologies are a very important tool in the study of convergence, namely in *hyperspaces*, that is, spaces of (closed) subsets of topological spaces, such as spaces of curves or continua, for example.

Dispensing with both reflexivity and transitivity, we here offer an ultimate generalization of topological spaces, **peritopological spaces**. Therein, a point does not necessarily belong to all its neighborhoods and, moreover, for some points x, all subsets might be neighborhoods, that is $\mathcal{V}(x)$ can be the *unproper* filter.

The blend of logic and topology started very early, with the origins of both. The topological model, for instance, was a great help to the comprehension of negation in intuitionism. Though topological models in modal logic, and the corresponding notion of bisimulation, are now well-known, to our knowledge, this is the first **systematic** introduction of peritopology (and pretopology) as such, in modal logic. It is, by no means, an artificial *contraption*. That points may not necessarily belong to their neighborhoods is an essential feature which can lead to many interesting interpretations.

An Alexandroff topological space is one in which every point has a *smallest* neighborhood. Those spaces are in exact correspondence with **preorders**, that is, binary relations which are, both, reflexive and transitive. The definition extends to peritopological (and pretopological) spaces. So that, Alexandroff pretopological spaces are in exact correspondence with reflexive binary relations, and Alexandroff peritopological spaces, simply, with all binary relations.

To end this introduction, let us just mention the following: Whereas, on a given set E, topologies can be viewed as **closed preorders** on the Stone space $\beta(E)$ of ultrafilters on E and pretopologies as **closed reflexive** binary relations on $\beta(E)$, peritopologies are easily identified to **closed** binary relations on $\beta(E)$.

But this is another story...

2. Peritopological Spaces

Generalizing topologies, G. Choquet [5] introduced *pretopologies*, using many different (though equivalent) definitions. We, first, recall two of them.

2.1 Pretopology

A pretopology on a given set E is defined to be a family $(\mathcal{V}(x))_{x\in E}$ of filters $\mathcal{V}(x)$ on E such that $x \in V$ for each $V \in \mathcal{V}(x)$. The elements $V \in \mathcal{V}(x)$ are called neighbourhoods of the point x for this pretopology. A pretopological space is a set with a pretopology on it.

A preclosure operation on E is an operation $g : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$ on subsets of E having the following three properties. For any subsets X and Y of E,

(i) $g(\emptyset) = \emptyset$,

(ii) $g(X \bigcup Y) = g(X) \bigcup g(Y)$,

(iii) $X \subset g(X)$

Notice that, from condition (ii), easily follows condition

(iv) $X \subset Y$ implies $g(X) \subset g(Y)$.

That preclosures and pretopologies are *dual*, two faces of the same notion, can be seen very much the same as for closures and topologies. Indeed, given any subset X in a pretopological space, define X^- to be the set of points x such that each neighborhood V of x meets X. Then $X \longmapsto X^-$ is a preclosure operation associated to the pretopology. Conversely, start from a preclosure operation g on E and set $\mathcal{V}(x) = \{X : x \notin g(X^c)\}$ where X^c is the complement of X. Then, $(\mathcal{V}(x))_{x\in E}$ is a pretopology whose associated preclosure is nothing else but g, the preclosure one started with.

Now, just a slight change in the definitions of those structures leads to spaces where neighbourhoods of a point need not contain that point, and some points might even have an empty neighborhood. Indeed, deleting some of the conditions on pretopologies and preclosures, we introduce the following generalizations.

2.2 Peritopology

A peritopology on a given set E is defined to be a family $(\mathcal{V}(x))_{x\in E}$ where each $\mathcal{V}(x)$ is either a (proper) filter on E or $\mathcal{V}(x) = \mathcal{P}(E)$ (the unproper filter on E). The elements $V \in \mathcal{V}(x)$ are called neighbourhoods of the point x for this peritopology. A peritopological space is a set with a peritopology on it.

A periclosure operation on E is an operation $g : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$ having the following two properties:

(i) $g(\emptyset) = \emptyset$,

(ii) $g(X \bigcup Y) = g(X) \bigcup g(Y)$ for any subsets X and Y.

Clearly enough, periclosures and peritopologies are *dual*, quite as are preclosures and pretopologies. Given any subset X in a peritopological space, its periclosure is X^-, the set of points x such that each neighborhood V of x meets X. Conversely, given a periclosure operation g on E, the neighborhood filter of a point x in the associated peritopology is $\mathcal{V}(x) = \{X : x \notin g(X^c)\}$. For more details, see [7].

Some notions pertaining to general topological (and pretopological) spaces have their natural analogues for peritopological spaces. Such are the notions of **continuity**, **hemi-open relations**, **quotients**, **separation**, and **reduction**. Proofs are omitted when they follow the same lines as for topology and pretopology.

Let $(E, (\mathcal{V}(x))_{x\in E})$ and $(F, (\mathcal{W}(x))_{x\in F})$ be two given peritopological spaces.

2.3 Countinuity

A function $f : E \longrightarrow F$ is said to be continuous at a point $x \in E$ whenever $f^{-1}(W)$ is a neighborhood of x for each neighborhood $W \in \mathcal{W}(f(x))$ of $f(x)$. When f is continuous at every point, it is said to be a continuous function.

We will need the following observation, in the sequel.

2.4 A remark

Let $R \subset E \times F$ be a (binary) relation between sets E and F. Then, for each subset $X \subset E$, the following holds:

$R^{-1}(F \setminus R(X)) \subset E \setminus X$ (that is $R^{-1}(R(X)^c) \subset X^c$) as well as $R^{-1}(R(X)^c)^c \subset X$.

Indeed, assume that $x \notin X^c$. We have to show that $x \notin R^{-1}(R(X)^c)$. Now, we have $x \in X$, hence $R(x) \subset R(X)$, which amounts to say that $R(x) \bigcap R(X)^c = \emptyset$. Therefore $x \notin R^{-1}(R(X)^c)$. □

2.5 Hemi-open relations

Extending the class of open functions to binary relations between topological spaces E and F, Choquet [5] introduced the notion of "relation mi-ouverte" (hemi-open) : The relation $R \subset E \times F$ is said to be *mi-ouverte* whenever $R(U)$ is open in F for each open set U in E. Say that R is *bi-ouverte* when both, R and its converse R^{-1}, are mi-ouvertes.

To be used in bisimulation, here is our generalization of this notion to peritopology. Let $R \subset E \times F$ be a relation between the two peritopological spaces E and F. Define R to be adequate whenever, for each neigbourhood V of a point x in E, the subset $R(V)$ is a neigbourhood in F of each point $y \in R(x)$. Call R bi-adequate when both, R and its converse R^{-1}, are adequate.

Notice that, when R is bi-adequate then, for subsets $X \subset E$ and $Y \subset F$, we always have

$$R(X^{c-c}) \subset R(X)^{c-c},\ R^{-1}(Y^{c-c}) \subset R^{-1}(Y)^{c-c}. \quad (1)$$

Indeed, Let $y \in R(X^{c-c})$. Then there exists $x \in X^{c-c}$ such that xRy, which implies that $x \notin X^{c-}$, so that there exists $V \in \mathcal{V}(x)$ such that $V \cap X^c = \emptyset$. Making use of the remark above, we have $V \cap R^{-1}(R(X)^c) = \emptyset$, and hence $R(V) \cap R(X)^c = \emptyset$. It follows that $R(V) \in \mathcal{V}(y)$, since R is adequate and xRy. Therefore, $y \notin R(X)^{c-}$, and $y \in R(X)^{c-c}$ follows. The second inclusion is proved similarly. □

2.6 Separation

We, now, extend separation axioms from topology to peritopology.

The peritopological space E is said to be T_0 whenever each distinct points $x \neq y$ have *distinct* neigbourhood filters, i.e. $\mathcal{V}(x) \neq \mathcal{V}(y)$.

It is said to be T_1 whenever each distinct points $x \neq y$ have *uncomparable* neigbourhood filters, i.e. $\mathcal{V}(x) \not\subseteq \mathcal{V}(y)$ and $\mathcal{V}(y) \not\subseteq \mathcal{V}(x)$.

It is said to be T_2 whenever each distinct points $x \neq y$ have *incompatible* neigbourhood filters, i.e. no proper filter contains both $\mathcal{V}(x)$ and $\mathcal{V}(y)$.

2.7 Quotients

Let R be an equivalence relation on the peritopological space E and $p : E \longrightarrow E/R$ be the canonical projection. The *quotient peritopology* on E/R is defined by setting

$\mathcal{W}(p(x)) = \{V \subset E/R : p^{-1}(V)$ belongs to each $\mathcal{V}(y)$ such that $p(x) = p(y)\}$

Indeed, we have the following result whose proof is much the same as in ordinary topology.
The set $\mathcal{W}(p(x))$ is a (proper or unproper) filter on E/R and $(\mathcal{W}(p(x)))_{x\in E}$ is the maximal peritopology on E/R for which p is continuous.

2.8 Reduction

Define the relation $\leftrightsquigarrow$ on the peritopological space $(E, (\mathcal{V}(x))_{x\in E})$ by $x \leftrightsquigarrow y$ iff $\mathcal{V}(x) = \mathcal{V}(y)$. It is clear that $\leftrightsquigarrow$ is an equivalence relation on E.

The quotient peritopological space $E/\leftrightsquigarrow$ is called *the reduced space* , the projection $E \longrightarrow E/\leftrightsquigarrow$ being called *the reduction map.*

While the reduced space of a topological space is always a Kolmogorov space, this is not the case for peritopological spaces as the following example shows.

Example 2.1. Take two (finite or infinite) disjoint sets A and B with two distinguished elements $a \in A$ and $b \in B$, and let $E = A \bigcup B$. Then define

$\mathcal{F} = \{V \subseteq E : b \in V \supset A\}$, $\mathcal{G} = \{V \subseteq E : a \in V \supset B\}$

$\mathcal{V}(x) = \mathcal{F}$ for each $x \in A$, $\mathcal{V}(y) = \mathcal{G}$ for each $y \in B$

Clearly, $\mathcal{F}$ and $\mathcal{G}$ are filters on E, $\mathcal{V}(x)$'s and $\mathcal{V}(y)$'s define a peritopology on E, (indeed, a pretopology). The associated equivalence relation has two equivalence classes, A and B. The reduced space is not T_0.

3. Peritopological Models and Bisimulation

We present a generalization of the well-known topological model notion, and of the so-called topo-bisimulation to the peritopological setting. Both, also apply, of course, to the special pretopological case. For details, see [7].

Given a collection of proposition letters P, a *peritopological model* is a couple (E, μ) where E is a peritopological space and $\mu : P \longrightarrow \mathcal{P}(E)$ a valuation. Truth relation of a formula is defined as usual. The truth-set of φ in (E, μ) is denoted by $\mu(\varphi)$. Clearly then, $\mu(\Diamond\varphi) = \mu(\varphi)^-$ is the periclosure of $\mu(\varphi)$ and $\mu(\Box\varphi) = \mu(\varphi))^{c-c}$.

Basic notions of modal logic and bisimulation concept can be found in [4,7].

3.1 Bisimulation

A bisimulation between two peritopological models (E, μ) and (F, ν) is defined to be a nonempty bi-adequate relation $R \subset E \times F$ between the peritopological spaces E and F such that, for xRy, we always have $x \in \mu(p)$ iff $y \in \nu(p)$.

Clearly enough, pretopological models are a generalization of the notion of reflexive models for (basic) modal logic, and bisimulation between pretopological models is a generalization of the notion of bisimulation between reflexive models.

It is easily seen that the topological model notion is a special case of pretopological model. Furthermore, it is obvious that there is a one-to-one correspondence between Alexandroff topological spaces and preorders. Here, the concept of topo-bisimulation is presented in a different disguise, and we shall, readily, show that it is a special case of peritopological bisimulation. For a classical approach, see [1-3].

3.2 A step back (for a characterization)

Recall that a *topo-bisimulation* between two topological models (E, μ) and (F, ν) is a nonempty bi-open relation R between the spaces E and F satisfying the following condition:

$R(\mu(p)) \subset \nu(p)$ and $R^{-1}(\nu(p)) \subset \mu(p)$ for each $p \in P$.

The following **characterization** suffices to show that topo-bisimulation is, indeed, a special case of peritopological bisimulation.

Given a hemi-open relation R between two topological spaces, $(E, \mathcal{O}(E))$ and $(F, \mathcal{O}(F))$, the following condition holds: For each point $x \in E$ and each neigbourhood U [not neccessarily open] of x in the space E, the subset $R(U)$ is a neighboorhood of each point $y \in R(x)$. Conversely, a relation R between E and F for which this condition holds is hemi-open.

Proof. The necessity of the condition is obvious. As to sufficient condition, let $U \in \mathcal{O}(E)$. In order to show that $R(U) \in \mathcal{O}(F)$, it suffices to see that $R(U) \subset R(U)^\circ$ where $\circ$ denote the interior operator. Let $y \notin R(U)^\circ$. So, we have $y \notin R(U)^{c-c}$, hence $y \in R(U)^{c-}$. Then, we have $R(U)^c \cap V \neq \emptyset$ for each $V \in \mathcal{V}(y)$. Now assume that $y \in R(U)$. Then there exists $x \in U$ such that xRy, and U being an open subset of E, we have that U is a neighbourhood of x. But $R(U) \in \mathcal{V}(y)$ by hypothesis. Then from $R(U)^c \cap V \neq \emptyset$ for $V \in \mathcal{V}(y)$ we obtain $R(U)^c \cap R(U) \neq \emptyset$, which is a contradiction. So the hypothesis $y \in R(U)$ is not true. Therefore, y must not belong to $R(U)$. □

Theorem 3.1. *Let R be a bisimulation between two peritopological models (E, μ) and (F, ν). For any formula φ, if xRy then $x \in \mu(\varphi)$ iff $y \in \nu(\varphi)$.*

Proof. We use induction on the complexity of the formula φ. Atomic and Boolean cases for φ are straightforward. Consider the box case : $\varphi = \Box\psi$. Then the following computation is clear.

$$R(\mu(\Box\psi)) = R(\mu(\psi)^{c-c}) \subset R(\mu(\psi))^{c-c} \quad \text{by (1)}$$

So, by the induction hypothesis, $R(\mu(\psi))^{c-c} \subset \nu(\psi)^{c-c} = \nu(\Box\psi)$.

Conversely, we have

$$R^{-1}(\nu(\Box\psi)) = R^{-1}(\nu(\psi)^{c-c}) \subset R^{-1}(\nu(\psi))^{c-c} \quad \text{by (1) again.}$$

Thus, by the induction hypothesis, again, $R^{-1}(\nu(\psi))^{c-c} \subset \mu(\psi)^{c-c} = \mu(\Box\psi)$. □

In order to determine the logic of all peritopological spaces, we need the following result.

Lemma 3.2. *Let $\mathcal{F} = (E, R)$ be a frame and $\mathcal{A} = (E, (\mathcal{V}(x))_{x\in E})$ the corresponding Alexandroff peritopological space. Then given an arbitrary valuation μ on E, we have*

$$(\mathcal{F}, \mu), x \models \varphi \quad \textit{iff} \quad (\mathcal{A}, \mu), x \models \varphi$$

for any $x \in E$ and any modal formula φ.

Proof. Again, use induction on the complexity of φ, and consider only the modal case $\varphi = \Diamond\psi$, since Boolean cases are straightforward. We have

$$(\mathcal{F}, \mu), x \models \Diamond\psi \quad \text{iff} \quad (\mathcal{F}, \mu), y \models \psi \quad \text{for some } y \in R(x).$$

Since every $V \in \mathcal{V}(x)$ contains y, we obtain $(\mathcal{A}, \mu), x \models \Diamond\psi$ by the definition of $\models$.

Conversely, $(\mathcal{A}, \mu), x \models \Diamond\psi$ iff $(\mathcal{A}, \mu), y \models \psi$ for every $V \in \mathcal{V}(x)$ and for some $y \in V$. Now the definition of $\mathcal{V}(x)$ gives $R(x) \in \mathcal{V}(x)$, and that of $\models$ yields $(\mathcal{F}, \mu), x \models \varphi$. □

Theorem 3.3. *With respect to the class of all peritopological spaces, the minimal logic* **K** *is*

(i) sound,

(ii)strongly complete.

Proof. (i) Indeed, all axioms of **K** are valid in any peritopological space, and its inference rules preserve validity; see [7].

(ii) Let Σ be a set of **K**-consistent formulas. Since **K** is strongly complete with respect to the class of all frames, there is a frame $\mathcal{F}$ such that Σ is satisfiable on the model $\mathcal{M} = (\mathcal{F}, \mu)$. Now let $(\mathcal{A}, \mu)$ be peritopologic model where $\mathcal{A}$ corresponds to the frame $\mathcal{F}$ and μ the valuation in the model $\mathcal{M}$. Then Σ is satisfiable in $(\mathcal{A}, \mu)$ according to Lemma 3.2. $\Box$

Thus, the minimal logic K is the logic of all peritopological spaces.

4. The Logics of Some Classes of Peritopological spaces

Theorem 4.1. *The class of all T_0 -peritopological spaces is not modal definable.*

Proof. Assume that there is a modal formula φ which characterizes those T_0- peritopological spaces. Let E be a set with more than two elements. Choose two elements a and b, then define

$\mathcal{V}(a) := \{V \subseteq E : b \in V\}$

and for each $x \in E$ different from a,

$\mathcal{V}(x) := \{V \subseteq E : a \in V\}$.

It is clear that the peritopological space thus defined is not a T_0 - space. However, the reduced space $E/\leftrightsquigarrow$ is a T_0 - space. Moreover, the graph of reduction map $E \longrightarrow E/\leftrightsquigarrow$ is a peritopological-bisimulation. Thus, φ is refuted on E and by bisimulation it must be refuted on the reduced space as well, a contradiction. Therefore our assumption does not hold, i.e. the theorem is proved. $\Box$

That the class of all T_1 -peritopological spaces and the class of all T_2 -peritopological spaces are not modal definable can be poven as above.

4.1 Proper spaces

For a subset X in a given peritopological space E, define *the peri-interior* of X to be the subset X^{c-c}. This is a substitute to the interior in topological spaces.

Recall that for a subset X in a topological space we always have $Int(A) \subset Cl(A)$, where Int and Cl denote the interior and the closure operators, respectively. Moreover, $Int(A) = Cl(A^c)^c$ holds. However, this is not always the case for peritopological spaces, as illustrated by the following example.

Take the peritopological space $E = \{a, b, c\}$ with $\mathcal{V}(a) = \{V : \{b, c\} \subseteq V\}$, $\mathcal{V}(b) = \mathcal{V}(c) = \mathcal{P}(E)$. Let $X = \{a, b\} \subseteq E$. Then, $X^c = \{c\}$, $X^{c-} = \{a\}$ and $X^{c-c} = \{b, c\}$. But $X^- = \{a\}$, hence $X^{c-c} \nsubseteq X^-$.

In other words, in peritopological spaces the peri-closure of a set does not necessarily contain its peri-interior.

Call a peritopological space *proper* whenever the peri-closure of each subset X contains the peri-interior of X.

The modal logic **D** is known to be the extension of the minimal logic **K** with the axiom of the seriality $\Box p \to \Diamond p$.

We have the following two results.

Theorem 4.2. *(i) The logic* **D** *defines the class of proper peritopological spaces.*

(ii) It is sound and strongly complete with respect to the class of proper peritopological spaces.

Proof. (i) We only have to consider the formula $\Box p \to \Diamond p$ as **K** is the logic of all peritopolocical spaces. So we have to prove that $\Box p \to \Diamond p$ is valid in any peritopological space $\mathcal{T} = (E, (\mathcal{V}(x))_{x \in E})$ if and only if $\mathcal{T}$ is a *proper* space. But it is obvious, since for any set $X \subseteq E$ and under the periotopological interpretations of modal operators, $\Box p \to \Diamond p$ is nothing but $X^{c-c} \subseteq X^-$.

(ii) Soundness follows from (i). To show the completeness, we will use Kripke completeness of **D**. Let Σ be a set of **D**-consistent formulas. Since **D** is strongly complete with respect to the class of serial frames, there is a frame $\mathcal{F}$ such that Σ is satisfiable on the model $\mathcal{M} = (\mathcal{F}, \mu)$. Now consider the peritopological model $(\mathcal{A}, \mu)$ where $\mathcal{A}$ is the corresponding peritopological space to $\mathcal{F}$ and μ is the valuation on $\mathcal{M}$. Then by by Lemma 3.2, and the fact that $\mathcal{F}$ is a serial frame, $\mathcal{A}$ is a proper space, so Σ is satisfied on $(\mathcal{A}, \mu)$ by the same lemma . $\Box$

5. Syntax and semantics of $\mathcal{H}$ and $\mathcal{H}(@)$

Let PROP be a countably infinite set of proposition letters and NOM a countably infinite set of nominals, disjoint from the set PROP. Then the syntax of the languages $\mathcal{H}$ and $\mathcal{H}(@)$ is defined as follows:

$$\text{WFF} := \top \mid p \mid i \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \qquad (\mathcal{H})$$

$$\text{WFF} := \top \mid p \mid i \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi \mid @_i\varphi \qquad (\mathcal{H}(@))$$

where $p \in$PROP and $i \in$ NOM.

A *hybrid peritopological model* is a couple (E, μ) where E is a peritopological space and μ :PROP $\cup$ NOM $\longrightarrow \mathcal{P}(E)$ a valuation which sends propositional letters to subset of E and nominals to singleton sets of E . The semantics for $\mathcal{H}$ and $\mathcal{H}(@)$ is the same as for the basic modal language for the propositional letters, nominals, Boolean connectives, and the modality $\Diamond$. The semantic of @ as follows: for any $x, y \in E$ and any modal formula φ

$$(E, \mu), x \models @_i\varphi \quad \text{iff} \quad (E, \mu), y \models \varphi \text{ for } \mu(i) = \{y\} .$$

Validity and satisfiability with respect to a peritopolocigal space or class of peritopolocigal space is defined as for modal formulas. For details, see [7-9].

6. Bisimulation

$\mathcal{H}$ - *bisimulation* between two peritopological models (E, μ) and (F, ν) is defined to be simply a nonempty bi-adequate relation $R \subset E \times F$ between the peritopological spaces E and F such that, for xRy and all $p \in$PROP $\cup$ NOM, we always have $x \in \mu(p)$ iff $y \in \nu(p)$.
An $\mathcal{H}(@)$ - *bisimulation* is a $\mathcal{H}$ - bisimulation R satisfying in addition

If $x \in \mu(i)$ and $y \in \nu(i)$ for some $i \in$ NOM, then xRy.

Theorem 6.1. *Let $\mathcal{L}$ be one languages of $\mathcal{H}$ and $\mathcal{H}(@)$. R be a $\mathcal{L}$ - bisimulation between two peritopological models (E, μ) and (F, ν). For any formulaφ, if xRy then $x \in \mu(\varphi)$ iff $y \in \nu(\varphi)$.*

Proof. The proof is a straightforward generalization of proof given for the basic modal language. It suffices to show the following condition only for satisfaction operator:

$\mu(i) \in \mu(\varphi)$ iff $\nu(i) \in \nu(\varphi)$

Let $\mu(i) \in \mu(\varphi)$. Then, $R(\mu(i)) \subset R(\mu(\varphi))$ and since R is a bisimulation, we obtain $\nu(i) \in R(\mu(i))$. By the induction hypothesis, $R(\mu(\varphi)) \subset \nu(\varphi)$ and so, $\nu(i) \in \nu(\varphi)$.

Other direction is shown similarly. □

While the separation axioms T_0 and T_1 for topological spaces are not definable by the basic modal language, we have the following result for $\mathcal{H}(@)$ language :

Theorem 6.2. *Consider the following formulas in $\mathcal{H}(@)$:*

$$t_0 = @_i\neg j \to @_j\Box\neg i \vee @_i\Box\neg j,$$
$$t_1 = i \longleftrightarrow \Diamond i$$

(i) The topologic space $\mathcal{T} = (E, \mathcal{O}(E))$ is a T_0-space iff $\mathcal{T} \models t_0$

(ii) The topologic space $\mathcal{T} = (E, \mathcal{O}(E))$ is a T_1-space iff $\mathcal{T} \models t_1$

Proof. See [6] □

We also know that axioms T_0 and T_1 are not definable for peritopological spaces in the basic modal language. Are they definable in $\mathcal{H}(@)$? In particular, do formulas t_0 and t_1 suffice for peritopological definability?

Example 6.3. Let $E = \mathbb{Z} - \{0\}$. Let $\mathcal{V}(n) := \{V \subseteq E : \{-n\} \subseteq V\}$ for every $n \in E$. Then, it is clear that the peritopological spaces $\mathcal{T} = (E, (\mathcal{V}(n))_{n\in E})$ is a T_0-space . Let μ be any valuation defined the spaces by: $\mu(i) = \{n_0\}$ and $\mu(j) = \{-n_0\}$ for some $n_0 \in E$. Thus we have $(\mathcal{T}, \mu), n \models @_i\neg j$ for any $n \in E$. But one can easily verify the following:

$(\mathcal{T}, \mu), n_0 \not\models \Box\neg j$, $(\mathcal{T}, \mu), n \not\models @_i\Box\neg j$, $(\mathcal{T}, \mu), -n_0 \not\models \Box\neg i$ and $(\mathcal{T}, \mu), n \not\models @_j\Box\neg i$. Thus, $(\mathcal{T}, \mu), n \not\models @_j\Box\neg i \vee @_i\Box\neg j$ and hence $\mathcal{T} \not\models t_0$.

Let us see whether the converse is true.

Example 6.4. Let $E = \{x, y, z\}$ and $\mathcal{V}(x) = \mathcal{V}(y) = \mathcal{V}(z) := \{V : \{z\} \subseteq V \subseteq E\}$. Consider the peritopological spaces $\mathcal{T} = (E, (\mathcal{V}(x))_{x\in E})$

and show that $\mathcal{T} \models t_0$. Assume: $\mathcal{T} \not\models t_0$. There are a valuation μ and a point x such that $(\mathcal{T}, \mu), x \models @_i\neg j$ but $(\mathcal{T}, \mu), x \not\models @_j\Box\neg i \vee @_i\Box\neg j$. Then we have $(\mathcal{T}, \mu), \alpha \models \Diamond j$ and $(\mathcal{T}, \mu), \beta \models \Diamond i$ where $\mu(i) = \alpha$ and $\mu(j) = \beta$. Thus, in the model $(\mathcal{T}, \mu)$ there are points α and β named by i and j such that each neighborhood of α contains β and each neighborhood of β contains α, which is not possible. Therefore our assumption is false. Thus we have at the same time a space $\mathcal{T}$ is not a T_0-space.

Let us look now at T_1-spaces .

Example 6.5. Consider peritopological space in Example 3.1: $\mathcal{T} = (E, (\mathcal{V}(n))_{n\in E})$, Which is clearly a T_1-space. On this spaces let μ be any valuation defined by $\mu(i) = \{1\}$. Obviously $(\mathcal{T}, \mu), 1 \models i$. Since $\{-1\} \in \mathcal{V}(1)$ and $1 \notin \{-1\}$, we have $(\mathcal{T}, \mu), 1 \not\models \Diamond i$. Therefore $\mathcal{T}$ is a T_1-space but $\mathcal{T} \not\models t_1$ dir.

Lemma 6.6. *Let $\mathcal{T} = (E, (\mathcal{V}(x))_{x\in E})$ any peritopological space. Then if $\mathcal{T} \models t_1$ then $\mathcal{T}$, T_1-space.*

Proof. If $\mathcal{T} \models t_1$ then for every $x \in E$ we have $\{x\}^- = \{x\}$. So take y distinct from x. Since $y \notin \{x\}^-$, there is a $V \in \mathcal{V}(y)$ such that $V \cap \{x\} = \emptyset$, i.e $x \notin V$. Hence we have $V \notin \mathcal{V}(x)$ since for each $U \in \mathcal{V}(x)$, $U \cap \{x\} \neq \emptyset$. Thus $\mathcal{V}(y) \nsubseteq \mathcal{V}(x)$. $\mathcal{V}(x) \nsubseteq \mathcal{V}(y)$ can be shown similarly. Thus $\mathcal{T}$ is a T_1-space. □

Question

It is an open question whether T_0 and T_1- peritopologic spaces are modal definable in $\mathcal{H}(@)$.

7. Conclusion

Keeping the spatial flavor of topology in modal logic, the language of peritopology can cope with general situations where reflexivity and transitivity are absent, it is hoped

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Ege University Department of Mathematics,
35040 Bornova-Izmir, Turkey.

ahmet.hamal@ege.edu.tr

Yasar University, Department of Mathematics,
35040 Bornova-Izmir, Turkey.

mehmet.terziler@yasar.edu.tr