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Hitoshi OMORI

AN AXIOMATIZATION OF WANSING'S EXPANSION OF NELSON'S LOGIC

A b s t r a c t. The present note offers an axiomatization for an expansion of Nelson's logic motivated by Heinrich Wansing which serves as a base logic for the framework of nonmonotonic reasoning considered by Dov Gabbay and Raymond Turner. We also show that the expansion of Wansing is not conservative over intuitionistic logic, but at least as strong as Jankov's logic.

1. Introduction

In [5], Heinrich Wansing observes some problems faced in the case of having intuitionistic logic, suggested by Dov Gabbay in [1], or Kleene's strong three-valued logic, suggested by Raymond Turner [4], as a base logic for a framework of nonmonotonic reasoning. In order to overcome the counterintuitive results, Wansing introduces an expansion of Nelson's logic

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(cf. [6, 2]), and shows that the new logic serves well as a base for the non-monotonic reasoning. The expansion involves a connective M where $\mathsf{M}A$ is read as "it is consistent to assume at this stage that A".

Based on these, the aim of this note is to give an axiomatization of the system which was left as an open problem in [5, p.52].

2. Semantics and proof theory

After setting up the language, we first present the semantics, and then turn to the proof theory.

Definition 2.1. The language $\mathcal{L}$ consists of a finite set $\{\bot, \sim, \mathsf{M}, \wedge, \vee, \rightarrow\}$ of propositional connectives and a countable set Prop of propositional variables which we denote by p, q, etc. Furthermore, we denote by Form the set of formulas defined as usual in $\mathcal{L}$. We denote a formula of $\mathcal{L}$ by A, B, C, etc. and a set of formulas of $\mathcal{L}$ by Γ, Δ, Σ, etc.

2.1 Semantics

Let us now state the semantics. Although Wansing's focus was on one of the Nelson's logics known as **N3** in the literature[1], we will take a little more general system $\mathbf{N4}^{\bot}$, introduced by Sergei Odintsov in [3], as the base system and add the consistency connective.

Definition 2.2. A model for the language $\mathcal{L}$ is a triple $\langle W, \leq, V\rangle$, where W is a non-empty set (of states); $\leq$ is a partial order on W; and $V : W \times \mathsf{Prop} \longrightarrow \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$ is an assignment of truth values to state-variable pairs with the condition that $i \in V(w_1, p)$ and $w_1 \leq w_2$ only if $i \in V(w_2, p)$ for all $p \in \mathsf{Prop}$, all $w_1, w_2 \in W$ and $i \in \{0, 1\}$. Valuations V are then extended to interpretations I to state-formula pairs by the following conditions:

- $I(w, p) = V(w, p)$,
- $1 \notin I(w, \bot)$,

[1] In [5], Wansing refers to the system as **N**, but here we will use the updated notation from later publications.

- $0 \in I(w, \bot)$,
- $1 \in I(w, {\sim}A)$ iff $0 \in I(w, A)$,
- $0 \in I(w, {\sim}A)$ iff $1 \in I(w, A)$,
- $1 \in I(w, A \wedge B)$ iff $1 \in I(w, A)$ and $1 \in I(w, B)$,
- $0 \in I(w, A \wedge B)$ iff $0 \in I(w, A)$ or $0 \in I(w, B)$,
- $1 \in I(w, A \vee B)$ iff $1 \in I(w, A)$ or $1 \in I(w, B)$,
- $0 \in I(w, A \vee B)$ iff $0 \in I(w, A)$ and $0 \in I(w, B)$,
- $1 \in I(w, A \to B)$ iff for all $x \in W$: if $w \leq x$ and $1 \in I(x, A)$ then $1 \in I(x, B)$,
- $0 \in I(w, A \to B)$ iff $1 \in I(w, A)$ and $0 \in I(w, B)$,
- $1 \in I(w, \mathsf{M}A)$ iff for some $x \in W$: $w \leq x$ and $1 \in I(x, A)$,
- $0 \in I(w, \mathsf{M}A)$ iff $0 \in I(w, A)$.

Finally, semantic consequence is now defined as follows:

$$\Sigma \models A \text{ iff for all models } \langle W, \leq, I \rangle, \text{ and for all } w \in I: \quad 1 \in I(w, A) \text{ if } 1 \in I(w, B) \text{ for all } B \in \Sigma.$$

Remark 2.3. If we eliminate the clause for M, then we obtain the semantics for the system $\mathbf{N4}^{\bot}$. Note also that two falsity conditions are considered for M in [5]. Based on the observation given by Wansing in [5, p.51], we take the simpler version.

2.2 Proof Theory

We now turn to the proof theory. Since Nelson's logic is presented in terms of a Hilbert-style calculus in [5], we will follow that path, and present some axioms for the new connective.

Definition 2.4. The system $\mathbf{N4}^{\perp}(\mathsf{Md})$ consists of the following axiom schemata and a rule of inference:

$$A \to (B \to A) \tag{Ax1}$$
$$(A \to (B \to C)) \to ((A \to B) \to (A \to C)) \tag{Ax2}$$
$$(A \wedge B) \to A \tag{Ax3}$$
$$(A \wedge B) \to B \tag{Ax4}$$
$$(C \to A) \to ((C \to B) \to (C \to (A \wedge B))) \tag{Ax5}$$
$$A \to (A \vee B) \tag{Ax6}$$
$$B \to (A \vee B) \tag{Ax7}$$
$$(A \to C) \to ((B \to C) \to ((A \vee B) \to C)) \tag{Ax8}$$
$$\sim\sim A \leftrightarrow A \tag{Ax9}$$
$$\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B) \tag{Ax10}$$
$$\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B) \tag{Ax11}$$
$$\sim(A \to B) \leftrightarrow (A \wedge \sim B) \tag{Ax12}$$
$$\perp \to A \tag{Ax13}$$
$$\sim\perp \tag{Ax14}$$
$$(\mathsf{M}A \wedge (A \to \perp)) \to B \tag{Ax15}$$
$$\mathsf{M}A \vee (A \to \perp) \tag{Ax16}$$
$$\sim\mathsf{M}A \leftrightarrow \sim A \tag{Ax17}$$
$$\frac{A \quad A \to B}{B} \tag{MP}$$

Following the usual convention, we define $A \leftrightarrow B$ as $(A \to B) \wedge (B \to A)$. Finally, we write $\Gamma \vdash A$ if there is a sequence of formulas $B_1, \ldots, B_n, A$, $n \geq 0$, such that every formula in the sequence $B_1, \ldots, B_n, A$ either (i) belongs to Γ; (ii) is an axiom of $\mathbf{N4}^{\perp}(\mathsf{Md})$; (iii) is obtained by (MP) from formulas preceding it in sequence.

Remark 2.5. The subsystem of $\mathbf{N4}^{\perp}(\mathsf{Md})$ consisting of axioms (Ax1) through (Ax8) together with the rule of inference (MP) is the positive intuitionistic logic. Moreover, if we eliminate the axioms related to M, i.e. axioms (Ax15), (Ax16) and (Ax17) from $\mathbf{N4}^{\perp}(\mathsf{Md})$, then we obtain the system $\mathbf{N4}^{\perp}$. Finally, the system $\mathbf{N3}$ is obtained from $\mathbf{N4}^{\perp}$ by simply adding $A \to (\sim A \to B)$ as an axiom.

Before turning to the soundness and completeness proofs, we note that the deduction theorem is provable for $\mathbf{N4}^{\perp}(\mathsf{Md})$.

Proposition 2.6. *For any* $\Gamma \cup \{A, B\} \subseteq \mathsf{Form}$, $\Gamma, A \vdash B$ *iff* $\Gamma \vdash A \to B$.

Proof. It can be proved in the usual manner in the presence of axioms (Ax1) and (Ax2), given that (MP) is the sole rule of inference. □

3. Soundness and completeness

3.1 Soundness

As usual, the soundness part is rather straightforward.

Theorem 3.1 (Soundness)**.** *For* $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, *if* $\Gamma \vdash A$ *then* $\Gamma \models A$.

Proof. By induction on the length of the proof. We here only check that the three axioms for M are all valid.

For (Ax15): For any $w \in W$, the following holds.

$1 \in I(w, \mathsf{M}A \wedge (A \to \perp))$
iff $1 \in I(w, \mathsf{M}A)$ and $1 \in I(A \to \perp)$
iff (for some $x \in W$: $w \leq x$ and $1 \in I(x, A)$)
and (for all $x \in W$: if $w \leq x$ and $1 \in I(x, A)$ then $1 \in I(x, \perp)$)
iff (for some $x \in W$: $w \leq x$ and $1 \in I(x, A)$)
and (for all $x \in W$: if $w \leq x$ then $1 \notin I(x, A)$)
iff (for some $x \in W$: $w \leq x$ and $1 \in I(x, A)$)
and not (for some $x \in W$: $w \leq x$ and $1 \in I(x, A)$)

Therefore, we obtain that $1 \notin I(w, \mathsf{M}A \wedge (A \to \perp))$, and thus $\models (\mathsf{M}A \wedge (A \to \perp)) \to B$.

For (Ax16): For any $w \in W$, the following holds.

$$\begin{aligned}
&1 \in I(w, \mathsf{M}A \vee (A \to \bot)) \\
&\quad \text{iff } 1 \in I(w, \mathsf{M}A) \text{ or } 1 \in I(A \to \bot) \\
&\quad \text{iff (for some } x \in W\text{: } w \leq x \text{ and } 1 \in I(x, A)) \\
&\qquad \text{or (for all } x \in W\text{: if } w \leq x \text{ and } 1 \in I(x, A) \text{ then } 1 \in I(x, \bot)) \\
&\quad \text{iff (for some } x \in W\text{: } w \leq x \text{ and } 1 \in I(x, A)) \\
&\qquad \text{or (for all } x \in W\text{: if } w \leq x \text{ then } 1 \notin I(x, A)) \\
&\quad \text{iff (for some } x \in W\text{: } w \leq x \text{ and } 1 \in I(x, A)) \\
&\qquad \text{or not (for some } x \in W\text{: } w \leq x \text{ and } 1 \in I(x, A))
\end{aligned}$$

Therefore, $\models \mathsf{M}A \vee (A \to \bot)$.
For (Ax17): For any $w \in W$, the following holds.

$$\begin{aligned}
1 \in I(w, {\sim}\mathsf{M}A) &\text{ iff } 0 \in I(w, \mathsf{M}A) \\
&\text{ iff } 0 \in I(w, A)
\end{aligned}$$

Therefore, $\models {\sim}\mathsf{M}A \leftrightarrow {\sim}A$, and this completes the proof. □

3.2 Completeness

We now turn to the completeness proof. First, we introduce some standard notions.

Definition 3.2. A set of formulas, Σ, is *deductively closed* if the following holds:

$$\text{if } \Sigma \vdash A \text{ then } A \in \Sigma.$$

And Σ is *prime* if the following holds:

$$\text{if } A \vee B \in \Sigma \text{ then } A \in \Sigma \text{ or } B \in \Sigma.$$

Σ is *prime deductively closed* (pdc) if it is both. Finally, Σ is *non-trivial* if $A \notin \Sigma$ for some A.

Then the following two lemmas are well-known, and thus we will omit the details of the proofs. We only note in passing that the deduction theorem is the key for the second lemma.

Lemma 3.3. *If $\Sigma \nvdash A$ then there is a pdc, Δ, such that $\Sigma \subseteq \Delta$ and $\Delta \nvdash A$.*

Lemma 3.4. *If Σ is pdc and $A \to B \notin \Sigma$, there is a pdc Θ such that $\Sigma \subseteq \Theta$, $A \in \Theta$ and $B \notin \Theta$.*

Now, we are ready to prove the completeness.

Theorem 3.5 (Completeness)**.** *For $\Gamma \cup \{A\} \subseteq \mathsf{Form}$, if $\Gamma \models A$ then $\Gamma \vdash A$.*

Proof. We prove the contrapositive. Suppose that $\Gamma \nvdash A$. Then by Lemma 3.3, there is a $\Pi \supseteq \Gamma$ such that Π is a pdc and $A \notin \Pi$. Define the interpretation $\mathfrak{A} = \langle X, \leq, I\rangle$, where $X = \{\Delta : \Delta \text{ is a non-trivial pdc}\}$, $\Delta \leq \Sigma$ iff $\Delta \subseteq \Sigma$ and I is defined thus. For every state, Σ and propositional parameter, p:

$$1 \in I(\Sigma, p) \text{ iff } p \in \Sigma \text{ and } 0 \in I(\Sigma, p) \text{ iff } {\sim}p \in \Sigma$$

We show that this condition holds for any arbitrary formula, B:

$$1 \in I(\Sigma, B) \text{ iff } B \in \Sigma \text{ and } 0 \in I(\Sigma, B) \text{ iff } {\sim}B \in \Sigma \qquad (*)$$

It then follows that $\mathfrak{A}$ is a counter-model for the inference, and hence that $\Gamma \not\models A$. The proof of $(*)$ is by a simultaneous induction on the complexity of B with respect to the positive and the negative clause.

For bottom: For the positive clause, note that the semantic clause is $1 \notin I(\Sigma, \bot)$ and that (Ax13) together with the non-triviality of Σ gives us $\bot \notin \Sigma$. Therefore, we obviously have $1 \notin I(\Sigma, \bot)$ iff $\bot \notin \Sigma$, and so, by contraposition, the desired result is proved. For the negative clause, we use the semantic clause $0 \in I(\Sigma, \bot)$ as well as (Ax14).

For negation: We begin with the positive clause.

$$\begin{aligned} 1 \in I(\Sigma, {\sim}C) &\text{ iff } 0 \in I(\Sigma, C) \\ &\text{ iff } {\sim}C \in \Sigma && \text{IH} \end{aligned}$$

The negative clause is also straightforward.

$$\begin{aligned} 0 \in I(\Sigma, {\sim}C) &\text{ iff } 1 \in I(\Sigma, C) \\ &\text{ iff } C \in \Sigma && \text{IH} \\ &\text{ iff } {\sim}{\sim}C \in \Sigma && \text{(Ax9)} \end{aligned}$$

For disjunction: We begin with the positive clause.

$$\begin{aligned} 1 \in I(\Sigma, C \vee D) &\text{ iff } 1 \in I(\Sigma, C) \text{ or } 1 \in I(\Sigma, D) \\ &\text{ iff } C \in \Sigma \text{ or } D \in \Sigma && \text{IH} \\ &\text{ iff } C \vee D \in \Sigma && \Sigma \text{ is a prime theory} \end{aligned}$$

The negative clause is also straightforward.

$$\begin{aligned} 0 \in I(\Sigma, C \vee D) &\text{ iff } 0 \in I(\Sigma, C) \text{ and } 0 \in I(\Sigma, D) \\ &\text{ iff } {\sim}C \in \Sigma \text{ and } {\sim}D \in \Sigma && \text{IH} \\ &\text{ iff } {\sim}C \wedge {\sim}D \in \Sigma && \Sigma \text{ is a theory} \\ &\text{ iff } {\sim}(C \vee D) \in \Sigma && \text{(Ax11)} \end{aligned}$$

For conjunction: Similar to the case for disjunction, and thus we leave the details to the reader.
For implication: We begin with the positive clause.

$$\begin{aligned} 1 \in I(\Sigma, C \to D) &\text{ iff for all } \Delta \text{ s.t. } \Sigma \subseteq \Delta, \text{ if } 1 \in I(\Delta, C) \text{ then } 1 \in I(\Delta, D) \\ &\text{ iff for all } \Delta \text{ s.t. } \Sigma \subseteq \Delta, \text{ if } C \in \Delta \text{ then } D \in \Delta && \text{IH} \\ &\text{ iff } C \to D \in \Sigma \end{aligned}$$

For the last equivalence, assume $C \to D \in \Sigma$ and $C \in \Delta$ for any Δ such that $\Sigma \subseteq \Delta$. Then by $\Sigma \subseteq \Delta$ and $C \to D \in \Sigma$, we obtain $C \to D \in \Delta$. Therefore, we have $\Delta \vdash C \to D$, so by (MP), we obtain $\Delta \vdash D$, i.e. $D \in \Delta$, as desired. On the other hand, suppose $C \to D \notin \Sigma$. Then by Lemma 3.4, there is a $\Sigma' \supseteq \Sigma$ such that $C \in \Sigma'$, $D \notin \Sigma'$ and Σ' is a pdc. Furthermore, non-triviality of Σ' is obvious by $D \notin \Sigma'$. Thus, we obtain the desired result.

As for the negative clause, it is straightforward.

$$\begin{aligned} 0 \in I(\Sigma, C \to D) &\text{ iff } 1 \in I(\Sigma, C) \text{ and } 0 \in I(\Sigma, D) \\ &\text{ iff } C \in \Sigma \text{ and } {\sim}D \in \Sigma && \text{IH} \\ &\text{ iff } C \wedge {\sim}D \in \Sigma && \Sigma \text{ is a theory} \\ &\text{ iff } {\sim}(C \to D) \in \Sigma && \text{(Ax12)} \end{aligned}$$

For consistency: We begin with the positive clause.

$$\begin{aligned} 1 \in I(\Sigma, \mathsf{M}C) &\text{ iff for some } \Delta, \Sigma \subseteq \Delta \text{ and } 1 \in I(\Delta, C) \\ &\text{ iff for some } \Delta, \Sigma \subseteq \Delta \text{ and } C \in \Delta && \text{IH} \\ &\text{ iff } \mathsf{M}C \in \Sigma \end{aligned}$$

For the last equivalence, assume $\mathsf{M}C \in \Sigma$. Then, we have $(C \to \bot) \to \bot \in \Sigma$ by (Ax15), and since Σ is non-trivial, i.e. $\bot \notin \Sigma$, we obtain $C \to \bot \notin \Sigma$. Then by Lemma 3.4, there is a $\Sigma' \supseteq \Sigma$ such that $C \in \Sigma'$, $\bot \notin \Sigma'$ and Σ' is a pdc, as desired. For the other half, assume $\mathsf{M}C \notin \Sigma$ and $C \in \Delta$ for any Δ s.t. $\Sigma \subseteq \Delta$. Then by the former, (Ax16) and the primeness of Σ, we obtain $C \to \bot \in \Sigma$, and thus $C \to \bot \in \Delta$. This together with $C \in \Delta$ implies that $\bot \in \Delta$ which contradicts to the assumption that Δ is non-trivial.

As for the negative clause, it runs as follows.

$$\begin{aligned} 0 \in I(\Sigma, \mathsf{M}C) &\text{ iff } 0 \in I(\Sigma, C) && \\ &\text{ iff } {\sim}C \in \Sigma && \text{IH} \\ &\text{ iff } {\sim}\mathsf{M}C \in \Sigma && \text{(Ax17)} \end{aligned}$$

Thus, we obtain the desired result. □

Remark 3.6. For the axiomatization of the original case, we first eliminate $\bot$ from the language. Moreover, we need to make the following small changes. Semantically, we make the valuation to assign one of the values $\{1\}, \emptyset$ or $\{0\}$, but not $\{1, 0\}$, and we delete the clauses for $\bot$. Proof-theoretically, we eliminate axioms (Ax13) and (Ax14), add $A \to ({\sim}A \to B)$, and replace axioms (Ax15) and (Ax16) by the following respectively:

$$(\mathsf{M}A \wedge (A \to {\sim}A)) \to B \tag{Ax15'}$$

$$\mathsf{M}A \vee (A \to {\sim}A) \tag{Ax16'}$$

Then, the soundness and completeness will follow in the same manner.

4. Reflections and concluding remarks

Let us now briefly examine the system $\mathbf{N4}^{\bot}(\mathsf{Md})$. In the case of $\mathbf{N4}^{\bot}$, the unary operation $\neg A$, which is an abbreviation for $A \to \bot$, is exactly the intuitionistic negation. However, this is not the case anymore in $\mathbf{N4}^{\bot}(\mathsf{Md})$.

Proposition 4.1. *The following formulas are provable in* $\mathbf{N4}^{\bot}(\mathsf{Md})$.

$$(A \vee B) \to (\neg A \to B) \tag{1}$$

$$\mathsf{M}A \leftrightarrow \neg\neg A \tag{2}$$

$$\neg A \vee \neg\neg A \tag{3}$$

Proof. For (1), it is enough to prove $A \to (\neg A \to B)$ and $B \to (\neg A \to B)$, and these are obvious in $\mathbf{N4}^{\perp}(\mathsf{Md})$. For (2), the left-to-right direction is immediate in view of (Ax15), and the right-to-left direction follows by combining (Ax16) and (1). Finally, (3) is derivable in view of (2) and (Ax16). $\square$

Remark 4.2. Here are two remarks related to the above proposition. First, the provability of (3) shows that the addition of M makes the negation $\neg$ stronger than the intuitionistic negation. More specifically, the M-free fragment of $\mathbf{N4}^{\perp}(\mathsf{Md})$ is at least strong as Jankov's logic, the superintuitionistic logic which is semantically obtained by adding the directedness condition. Second, (2) may give the thought that M can be defined once we add the directedness condition. However, that is not the case as we have the following result.

Proposition 4.3. *The following formula is* not *provable in* $\mathbf{N4}^{\perp}(\mathsf{Md})$.

$$\sim\mathsf{M}A \leftrightarrow \sim\neg\neg A \tag{4}$$

Proof. If we have the above equivalence, then we obtain that $\sim A \leftrightarrow \neg A$ is provable in view of the equivalences $\sim\mathsf{M}A \leftrightarrow \sim A$ and $\sim\neg A \leftrightarrow A$. However, it follows that this is not the case by the following simple countermodel: take two states w, w' with $w \leq w'$, $I(w, p) = \{0\}$, and $I(w', p) = \{1\}$ for atomic p. Then we have $1 \in I(w, \sim p)$ and $1 \notin I(w, \neg p)$, and thus $1 \notin I(w, \sim p \leftrightarrow \neg p)$, as desired. $\square$

Remark 4.4. The derivability of (2) and non-derivability of (4) show that provable equivalence is *not* a congruence relation in $\mathbf{N4}^{\perp}(\mathsf{Md})$. As is well-known, this is also the case in $\mathbf{N4}$ and $\mathbf{N4}^{\perp}$ which can be observed by considering the axiom (Ax12). Indeed, $\sim(p \to q) \leftrightarrow (p \wedge \sim q)$ is derivable in $\mathbf{N4}$ and $\mathbf{N4}^{\perp}$, but $\sim\sim(p \to q) \leftrightarrow \sim(p \wedge \sim q)$, i.e. $(p \to q) \leftrightarrow (\sim p \vee q)$ is not.

In sum, we presented a Hilbert-style system which is sound and complete with respect to the semantics introduced by Wansing in [5]. Moreover, we observed that unlike Nelson's logics, $\mathbf{N4}^{\perp}(\mathsf{Md})$ is *not* conservative over intuitionistic logic, but at least as strong as Jankov's logic which is obtained by adding (3) to intuitionistic logic.

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The Graduate Center, City University of New York, USA
Currently: Department of Philosophy, Kyoto University, Japan
hitoshiomori@gmail.com
https://sites.google.com/site/hitoshiomori/home