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THERMAL CONDITIONS OF BUILDINGS: MATHEMATICAL MODELING BY POWER CIRCUIT THEORY

MODELOWANIE MATEMATYCZNE STANU CIEPLNEGO BUDYNKU METODAMI TEORII OBWODÓW ENERGETYCZNYCH

Abstract

The object of this study is to produce a mathematical model for heat conditions in a building, expressed as a complex power circuit. The building is characterized as a power circuit, complete with a detailed description of those components responsible for heat loss, using formal methods to make circuit calculations. Representing a building as a power circuit provides a powerful mathematical modelling tool for assessing heat processes (heat conditions) in separate rooms and in the building in a whole.

Keywords: *heat conditions*, *heat loss*, *power circuit*, *flow*, *effort*, *mathematical modelling*

Streszczenie

W niniejszym artykule przedstawiana jest budowa jako obwód energetyczny ze szczegóło-wym odtwarzaniem różnych składowych strat cieplnych budynku. Obliczania takiego obwodu są wykonywane formalizowanymi metodami. Odtwarzanie stanu cieplnego budynku przez ob-wód energetyczny daje potężny instrument modelowania matematycznego procesów cieplnych (stanu cieplnego) zarówno pomieszczeń pojedynczych, jak i budynku w całości.

Słowa kluczowe: *stan cieplny*, *straty cieplne*, *obwód energetyczny*, *strumień*, *natężenie*, *modelowanie matematyczne*

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1. Introduction

The modern building has a complex architectural and structural system with various spatial and engineering elements. The traditional method for calculating energy balance is to calculate the thermal conditions of the building [1, 2].The demand for heating must take account of the heat loss through the enclosure, infiltration of external air or conditions affecting heat loss as well as heat gains from solar energy, human and household activity.

The energy balance approach does not provide a tool for the analysis of complex air conditioning systems, such as heating, cooling and ventilation. To apply a modern approach, it is necessary to apply power circuit theory to our methods.

2. Simulating the thermal conditions of a building using power circuit theory in our methodology

Traditionally, calculating heat loss from a building had to take into account heat transfer through interior heated rooms [1, 2]:

$$P = F \frac{T_{in} - T_{out}}{R_{ht.s}} \tag{1}$$

where:

F – enclosurearea [m^2],
$R_{ht.s}$ – heat transfer resistance [m^2·K/W],
T_{out}, T_{in} – interior air and environmental temperatures.

In addition to heat loss caused by heat transfer through the building's interior, heat loss is also caused by infiltration (condition):

$$Q_i = ck_{ch}m(T_{in} - T_{out}) = ck_{ch}\,\rho V(T_{in} - T_{out}) \tag{2}$$

where:

c – specific heatcapacity of air, $c = 1.005$ kJ/(kg·K),
k_{ch} – number of air changer per hour, $k_{ch} = 1$/h,
ρ – air density, $\rho = 1.2255$ kg/m^3,
V – volume of heated room.

Applying the theory that a building should be represented by a power circuit, passive (resistive) and active components [3] are required. Resistive components reflect heat loss through the enclosed building spaces (their resistances are determined by heat transfer resistance $R_{ht.s}$) and air infiltration or deterioration in the condition of the building.
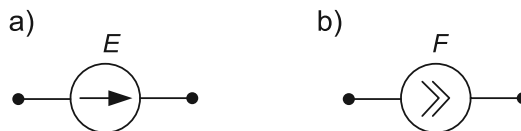

Fig. 1. Sources of heat energy: a) Source of Effort, b) Source of Flow

Sources of heat energy in power circuits are replaced by 'Source of Effort' *E* (Fig. 1a) or 'Source of Flow' *F* (Fig. 1b).

Source of effort can represent a heating system which provides a stable temperature inside the building. Source of flow can represent an appliance which provides a certain amount of heat for a building (e.g. fireplace).

3. Example of comparative calculation of building heat loss

The thermal conditions of a building with two rooms (Fig. 2) can be calculated. In particular, a heated room and an un-heated annex will be compared in terms of their efficiency according to the traditional method and the power circuit method.
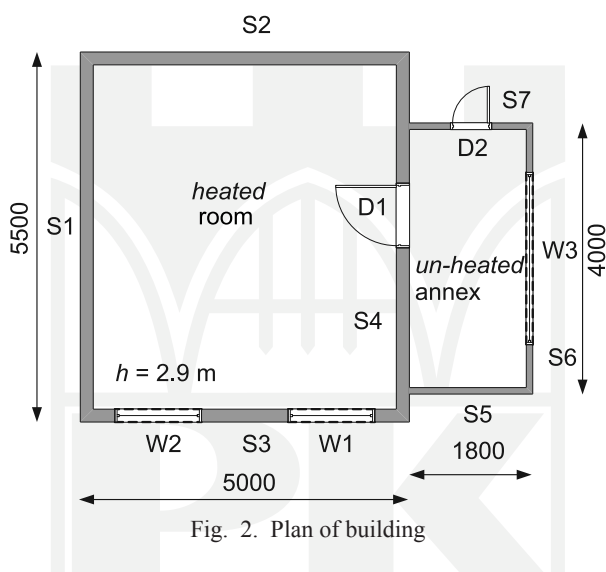

Fig. 2. Plan of building

We will consider heat loss through the following enclosures – walls (S1, S2, …, S7), windows (W1, W2, W3),doors (D1, D2) and air infiltration losses. The floor and ceiling are not included in this exercise.

3.1. Calculations using the heat balance method

According to [1, 2] heat loss from an enclosed heated room in a building is determined by equation (1).

If the heated room is connected to an un-heated annex, then heat loss, through enclosures separating these rooms are multiplied by an additional empirical factor *k* [1, 2]. Then, the equation (1) is expressed as:

$$P = F \frac{T_{in} - T_{out}}{R_{ht.s}} k \qquad (3)$$

Measurements and calculations for heated rooms of varying dimensions are shown in Table. 1. Here, we see the results of calculations for heat loss through surfaces applying equations(1) and (3).We assume that $T_{out} = -19$ºC, $T_{in} = 20$ºC.

The next step is to substitute numerical values (volume of a heated room $V = 79.8$ m³):

$$Q_i = 1.005 \cdot 1.2255 \cdot 79.8(20 - (-19)) = 3830.7 \text{ kJh,}$$

this equation corresponds toa power rating of 1064.1 W.

Table 1

Measurementsof heated rooms and heat loss calculations

Enclosures	Dimensions		F	$R_{ht.s}$	k	P
	a [m]	b [m]	[m²]	[m²·K/W]		[W]
S1	5.5	2.9	15.95	1.10		565.5
S2	5.0	2.9	14.50	1.10		514.1
S3	5.0	2.9	8.74	1.10		309.9
W1	1.6	1.8	2.88	0.42		267.4
W2	1.6	1.8	2.88	0.42		267.4
S4 (environment)	5.5	2.9	4.35	1.10		154.2
S4 (annex)			11.60	1.10	0.4	164.5
D1	1.4	2.5	3.50	0.64	0.4	85.3
Sum total						2328.3

Thus, the building's total heat loss expressed in terms of power is 3392.4 W.

3.2. The power circuit calculation method

In the simplest of cases, the circuit that is equivalent to a heated room is shown in Fig. 3. In this basic circuit, each enclosed space/room is represented by its equivalent resistive
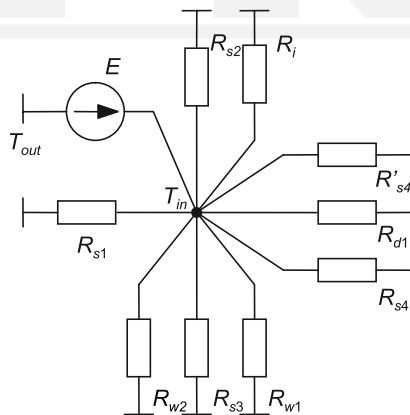

Fig. 3. Circuits representing a heated room

component. The index of the component corresponds to the description of the enclosed space/room. The wall (S4) is represented by two components: R_{s4} – as the part of the wall that is connected to the environment, and R'_{s4} – as the part of the wall that is connected to the un-heated annex. Heat loss through air infiltration is represented by component R_i.

The motive power of the equivalent circuit relates to the temperature difference between the air inside the building T_{in} and the environmental temperature T_{out}:

$$E = T_{in} - T_{out} \tag{4}$$

In theory, any power circuit can be described according to the following sets of equations [3, 4]:

$$\begin{cases} \mathbf{\Pi}\,\vec{X} = -\vec{F} \\ \mathbf{\Gamma}\,\vec{Y} = 0 \\ \vec{Y} = \vec{E} - \mathbf{R}\vec{X} \end{cases} \tag{5}$$

where:

$\mathbf{\Pi}$, $\mathbf{\Gamma}$ – incidence and circle matrixes,

$\vec{X}$, $\vec{Y}$ – column vectors of flows (serial variables) and efforts (parallel variables) of branches,

$\vec{E}$, $\vec{F}$ – column vectors of effort and flow sources,

$\mathbf{R}$ – resistence matrix of the resistive components.

The first equation in the system (the nodal equation) represents the compliance of the flows (serial variables) under Kirchhoff's first law. The second equation in the system (the circle equation) represents the compliance of efforts (the parallel variables) under Kirchhoff's second law. The third equation of the system (pole equation) depicts the connections between the main variables of the power circuit branches.

It is important to determine the actual terms of the main variables – flow X and effort Y. In engineering terms, heat is expressed using the temperature difference ΔT for effort Y, measured in K. Flow X takes heat flow q and is measured by W. Conveniently, this corresponds to the accepted methods for calculating heat loss in buildings [1, 2].

In this case, the measurement of resistance of the resistive components of the equivalent circuit is equal to K/W. The values are determined by the division of heat transfer resistance of the enclosed room by the area $R = R_{ht.s}/F$.

We apply the following equation to determine the resistance of the resistive component, which represents heat loss infiltration:

$$R_i = \frac{3.6}{cmk_{o1}} = \frac{3.6}{c\rho Vk_{o1}} \tag{6}$$

Similarly, we can consider the loss caused by air conditioning systems. The parameters of the resistive components of the equivalent circuit (Fig. 3) are shown in Table 2.

The motive effort of the source in compliance with (4) is the same as $E = 20 - (-19) = 39\,\text{K}$.

The parameters of the power circuit can be determined through direct equation calculations (5). However we can also use more effective methods, for example the nodal

voltage method. This proposed method is analogous to the nodal voltage method used in electrical engineering [5].

The results of calculations of the flows in circuit branches are shown in Table 2.

The total power of heat loss from the heated room is equal to the sum of the flows (serial variables) of all resistive components of the power circuit.

Table 2

Parameters of the components of the equivalent circuit ofa heatedroom and the parameters of the main mode

Resistive component	$R_{ht.s}$ [m²·K/W]	F [m²]	k	$R = R_{ht.s}/F$ [K/W]	X [W]
R_{s1}	1.10	15.95		0.069	565.5
R_{s2}	1.10	14.50		0.076	514.1
R_{s3}	1.10	8.74		0.126	309.9
R_{w1}	0.42	2.88		0.146	267.4
R_{w2}	0.42	2.88		0.146	267.4
R_{s4}	1.10	4.35		0.253	154.2
R'_{s4}	1.10	11.60	0.4	0.237	164.2
R_{d1}	0.64	3.50	0.4	0.457	85.3
R_i	–	–	–	0.037	1064.1
Sum total					3392.4

This example shows that the same results occur when applying both methods to the same elements of the building(see Table 1 and Table 2). However, the advantages of the power circuit theory became apparent after all of the relevant elements of the building were represented in more detail in the equivalent circuit. If all of the enclosed rooms in the building (both heated and unheated rooms) are represented in the equivalent circuit (Fig. 4), it follows that we should not use the empirical factor k (3).
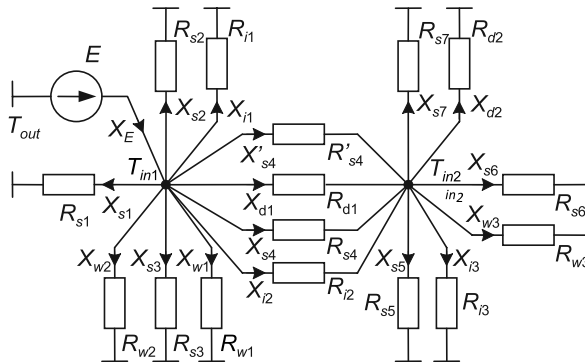

Fig. 4. Equivalent circuit of the building

The parameters of the resistive components of the equivalent circuit (Fig. 4) and the main mode parameters are shown in Table 3.

Parameters of the components of the equivalent circuit of the building and the main mode parameters

Resistive component	$R_{ht.s}$ [m²·K/W]	F [m²]	$R = R_{ht.s}/F$ [K/W]	X [W]	Y [K]
	1.10	15.95	0.0690	134.17	−24.54
R_{s2}	1.10	14.5	0.0759	565.28	−39.00
R_{s3}	1.10	8.74	0.1259	513.83	−39.00
R_{w1}	0.42	2.88	0.1458	309.77	−39.00
R_{w2}	0.42	2.88	0.1458	267.49	−39.00
R_{s4}	1.10	4.35	0.2529	267.49	−39.00
R'_{s4}	1.10	11.6	0.0948	953.55	−39.00
R_{d1}	0.64	3.5	0.1829	97.03	−24.54
R_{i1}			0.0409	258.86	−24.54
R_{i2}			0.3500	70.11	−24.54
R_{s5}	0.86	5.22	0.1648	87.75	−14.46
R_{s6}	0.86	6.2	0.1387	104.26	−14.46
R_{w3}	0.38	5.4	0.0704	205.41	−14.46
R_{s7}	0.86	2.97	0.2896	49.93	−14.46
R_{d2}	0.64	2.25	0.2844	50.85	−14.46
R_{i3}			0.2333	61.98	−14.46

In addition to calculating the flows and efforts of the circuit branch potentials (i.e. temperatures) all nodes of the power circuit (i.e. the rooms of the building) must be determined:

$$\vec{T} = eT^* + \mathbf{R}_\partial \vec{Y}_\partial \qquad (7)$$

where:

T^* – potential (temperature) of the basic node of power circuit,
e – unit vector,
$\mathbf{R}_\partial$ – path matrix,
$\vec{Y}_\partial$ – column vector of efforts of the linked circuitbranches.

According to (7), the temperature of an un-heated room is equal to −5.5°C. The calculation is not governed by the heat balance method.

It is important to note that, in this example, the total sum of the flow (i.e. serial variables) of all resistive components (7435 W) is greater than the heat flow source (3437.5 W).

On first examination, the balance between the power levels of the source and receiver is not maintained. This is explained due to an inconsistency between Trent's third condition and the selected measurements for flow [W] and effort [K]. According to [6], the measurement of flow (serial variable) and effort (parallel variable) must be physically compatible, in order to provide power:

$$P = XY \qquad (8)$$

This equation corresponds to the Bond Graph theory, the main principle of which is that power between two element combinations of is transferred by the combination of effort (parallel variable) and flow (serial variable) [7].

Therefore, heat loss in separate rooms can be determined as an algebraic sum of the flow of the resistive components which are incidental to the node of the placement in the equivalent circuit:

$$P_{np} = X_{s1} + X_{s2} + X_{s3} + X_{w1} + X_{w2} + X_{s4} + X'_{s4} + X_{d1} + X_{i1} + X_{i2} = 3437.52 \ \text{W}.$$

In order to determine the heat loss of the building, it is necessary to add the flow of the resistive components that are incidental to the environmental node (corresponding to the external facade of the building) to the algebraic formula.

4. Heat equivalent circuits of wall enclosure

Heat is transferred through the building's exterior enclosure by processes such as conduction, convection and radiation. A plan of heat transfer through a one-layered wall of a building [1] is shown in Fig. 5, where the lined dashess how the conditional margins for the air layer near the wall.
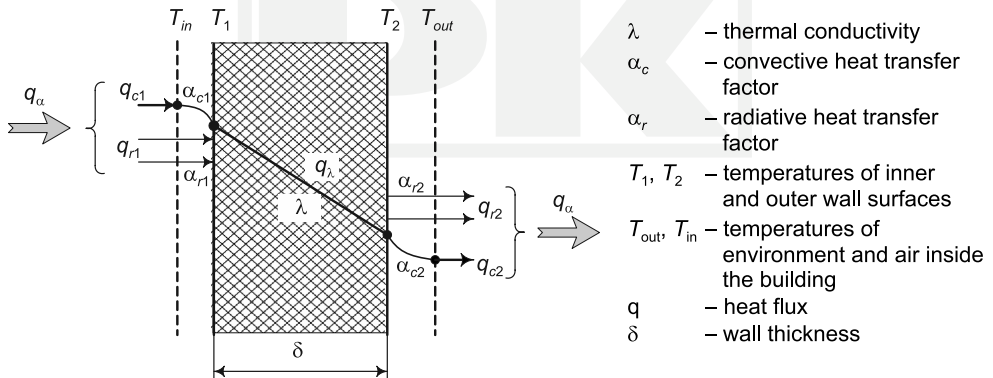

Fig. 5. Scheme ofheat transfer through a one-layered wall

The wall enclosure can be represented by heat equivalent circuits, of different specifications (Fig. 6).

Firstly, we must consider the mechanism of heat transfer between the wall enclosure and the air as well as the heat conductivity mechanism.
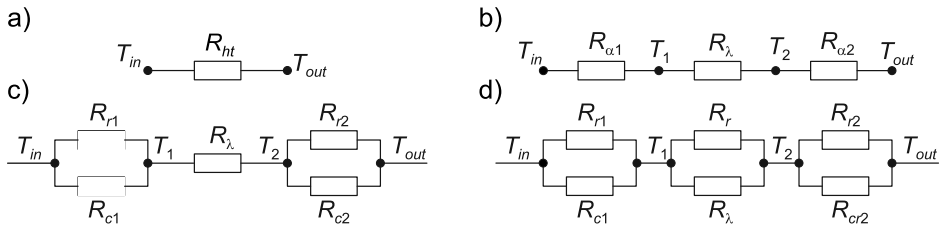
a)

b)

c)

d)

Fig. 6. Heat equivalent circuits of a one-layered wall

Static heat flux through a wall with thickness δ and the temperature difference between its surfaces $T_1 - T_2$ is calculated applying Fourier's equation:

$$q_\lambda = \lambda \frac{T_1 - T_2}{\delta} = \frac{T_1 - T_2}{R_\lambda} \tag{9}$$

where:
  λ – thermal conductivity of the wall material [W/(m·K)],
  δ – wall thickness [m].

Heat flux from the air inside the building to the wall enclosure $q_{\alpha 1}$ and then to the external environment $q_{\alpha 2}$ is equal to:

$$q_{\alpha 1} = \alpha_1(T_{in} - T_1), \quad q_{\alpha 2} = \alpha_2(T_2 - T_{out}) \tag{10}$$

where:
  α_1, α_2 – convective heat transfer factor on the internal air-wall boundary and wall-environment [W/(m²·K)].

The equivalent circuit for the wall enclosure (Fig. 6a) may be transformed as in Fig. 6b. The resistive components regarding the K/W relationship will be equal to:

$$R_{ht} = R_{\alpha 1} + R_\lambda + R_{\alpha 2} = \frac{1}{\alpha_1 F} + \frac{\delta}{\lambda F} + \frac{1}{\alpha_2 F} \tag{11}$$

where:
  R_{ht} – heat transfer resistive component [K/W],
  R_λ – thermal resistive component, $R_\lambda = \delta/(\lambda F)$ [K/W],
  $R_{\alpha 1}, R_{\alpha 2}$ – resistive components of heat transfer between the wall surface and the air inside and outside the building $R_{\alpha 1} = 1/(\alpha_1 F)$, $R_{\alpha 2} = 1/(\alpha_2 F)$ [K/W].

Heat transfer factors α on the margin between the wall and the air are divided into two components – convective α_c and radiative α_r:

$$\alpha_1 = \alpha_{r1} + \alpha_{c1}, \quad \alpha_2 = \alpha_{r2} + \alpha_{c2} \tag{12}$$

The equivalent heat transfer resistance circuit is shown in Fig. 2a, taking all components into account. The resistive components(radiative and convective) are defined as:

$$R_{r1} = \frac{1}{\alpha_{r1} F}, \quad R_{c1} = \frac{1}{\alpha_{c1} F}, \quad R_{r2} = \frac{1}{\alpha_{r2} F}, \quad R_{c2} = \frac{1}{\alpha_{c2} F} \tag{13}$$

The equivalent circuit (Fig. 6d) applies to transparent wall enclosures. Here, heat conduction takes place through radiation as well as by thermal conduction [1]. The corresponding resistive component R_r regarding the K/W relationship is equal to:

$$R_r = \frac{mk\delta}{16\sigma_0 n^2 \left(\dfrac{T_1 + T_2}{2}\right)^3} \frac{1}{F} \tag{14}$$

where:

m	–	opto-geometrical parameter,
k	–	attenuation coefficient [m^{-1}],
σ_0	–	Stefan–Boltzmann constant,
n	–	refractive index.

The thermal conditions of buildings, constructed of modern materials, can be analysed by the proposed, detailed equivalent circuit enclosure method.

5. The problem of measuring variables

In order to apply the general rule of power balance for the heat circuit, we must use another system of main variables. Accordingly [6] selecting main variables [4, 7] is recommended:
– serial variable (flow) – flow of entropy S [W/K],
– parallel variable (effort) – temperature differences ΔT [K].

In this case, the measurement of the resistance of the resistive component of the equivalent circuit is equal to K^2/W. However, the question of how to determine the resistance of the resistive component, using such a measurement, remains unsolved.

6. Conclusions

The application of power circuit theory methods for calculating the thermal conditions of a building provides a powerful tool for mathematical modeling of thermal processes (i.e. thermal conditions) both for separate rooms as for the building as a whole. A detailed representation of the building's wall enclosures and its heating sources allows for empirical factors to be discounted and for the closest possible approximations to be obtained in terms of the parameters of the building's actual and required thermal conditions.

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