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HILBERT ALGEBRAS WITH A NECESSITY MODAL OPERATOR

A b s t r a c t. We introduce the variety of Hilbert algebras with a modal operator $\Box$, called $H\Box$-algebras. The variety of $H\Box$-algebras is the algebraic counterpart of the $\{\rightarrow, \Box\}$-fragment of the intuitionitic modal logic $\mathbf{IntK}_{\Box}$. We will study the theory of representation and we will give a topological duality for the variety of $H\Box$-algebras. We are going to use these results to prove that the basic implicative modal logic $\mathbf{IntK}_{\Box}^{\rightarrow}$ and some axiomatic extensions are canonical. We shall also to determine the simple and subdirectly irreducible algebras in some subvarieties of $H\Box$-algebras.

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1. Introduction

We understand by an *intuitionistic modal logic* any subset of formulas in a propositional language $\mathcal{L}_m$ endowed with a set of unary modal operators M containing all the theorems of intuitionistic propositional logic **Int**, and closed under the rules of Modus Ponens, substitution and the regularity rule $\phi \to \alpha / m\phi \to m\alpha$, for each unary operator $m \in M$. In the literature exist several intuitionistic modal logics. There are logics with a necessity modal operator $\Box$, as the basic intuitionistic modal logic $\mathbf{IntK}_\Box$ (see [19] or [26]). Extensions of $\mathbf{IntK}_\Box$ was studied in [16], [19], [20], and [22]. Also we have a basic intuitionistic modal logic $\mathbf{IntK}_\Diamond$ in the language $\mathcal{L}_\Diamond$, and defined as the smallest logic to contains the axioms $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$ and $\neg\Diamond\bot$. Extensions of $\mathbf{IntK}_\Diamond$ was studied in [12], [19], [20], and [26]. We can also define a logic $\mathbf{IntK}_{\Box\Diamond}$, with the modal operators $\Box$ and $\Diamond$, as the smallest logic in the language $\mathcal{L}_{\Box\Diamond}$ containing both $\mathbf{IntK}_\Box$ and $\mathbf{IntK}_\Diamond$. Extensions of $\mathbf{IntK}_{\Box\Diamond}$ was studied in [1], [2], [14], [13], [19], and [20]. Just as Heyting algebras are the algebraic counterpart of **Int**, Heyting algebras with modal operators are the algebraic counterpart of the intuitionictic modal logics $\mathbf{IntK}_\Box$, $\mathbf{IntK}_\Diamond$ and $\mathbf{IntK}_{\Box\Diamond}$.

It is known that the variety Hil of Hilbert algebras is the algebraic semantic of the positive implicative fragment $\mathbf{Int}^{\to}$ of the intuitionistic propositional calculus **Int** (see [11], [18] or [24]). So, it is natural to ask for the implicative reducts of some intuitionistic modal logics. Again here we have multiple possibilities. For example, we can studied the fragments $\{\to, \Box\}$ and $\{\to, \vee, \Diamond\}$ of the intuitionistic modal logics $\mathbf{IntK}_\Box$ and $\mathbf{IntK}_\Diamond$, respectively. Another interesting possibility is to study some $\{\to, \vee, \Box, \Diamond\}$-fragments of $\mathbf{IntK}_{\Box\Diamond}$, or the intuitionitic modal logic $\mathbf{FS}_{\Box\Diamond}$ defined by Fischer-Servi in [14]. In this paper we will start studying the algebraic semantic of the $\{\to, \Box\}$-fragment of the intuitionistic normal modal logic $\mathbf{IntK}_\Box$. This fragment is denoted by $\mathbf{IntK}_\Box^{\to}$. The class of algebras associate with $\mathbf{IntK}_\Box^{\to}$ is the variety $\mathrm{Hil}_\Box$ of Hilbert algebras with a necessity modal operator $\Box$. We note that the variety of modal Tarski algebras studied in [5] is the algebraic semantics of the $\{\to, \Box\}$-fragment of the classical modal logic **K**, and thus is a subvariety of $\mathrm{Hil}_\Box$.

The paper is organized as follows. In Section 2 we will recall the definitions and some basic properties of Hilbert algebras and we will recall the topological representation and duality for Hilbert algebras developed in [9].

Also, we will recall the relational semantic of the implicational fragment of intuitionistic logic defined by R. Kirk in [21]. In Section 3 we will introduce the Hilbert algebras with a unary operator $\Box$, or $H\Box$-algebras for short. We will develop the topological representation and duality for $H\Box$-algebras using the simplified representation given in [9]. In Section 4 we shall characterize the $H\Box$-algebras that satisfy certain equations by means of first-order conditions defined in the dual space. Each of these varieties corresponds to an axiomatic extension of $\mathbf{IntK}_{\Box}^{\rightarrow}$. In Section 5 we will show that some implicational modal logics are canonical. Finally, in Section 6, we shall determine the simple and subdirectly irreducible algebras of some varieties of $H\Box$-algebras.

2. Preliminaries

In this section we will fix the terminology adopted in this paper.

Definition 2.1. *[11]* A Hilbert algebra is an algebra $A = \langle A, \rightarrow, 1\rangle$ of type $(2, 0)$ such that the following axioms hold in A:

1. $a \rightarrow a = 1$,
2. $1 \rightarrow a = a$,
3. $a \rightarrow (b \rightarrow c) = (a \rightarrow b) \rightarrow (a \rightarrow c)$,
4. $(a \rightarrow b) \rightarrow ((b \rightarrow a) \rightarrow a) = (b \rightarrow a) \rightarrow ((a \rightarrow b) \rightarrow b)$.

The variety of Hilbert algebras is denoted by Hil. It is easy to see that the binary relation $\leq$ defined in a Hilbert algebra A by $a \leq b$ if and only if $a \rightarrow b = 1$ is a partial order on A with greatest element 1.

Given a Hilbert algebra A and a sequence $a, a_1, \ldots, a_n \in A$, we define:

$$(a_1, \ldots, a_n; a) = \begin{cases} a_1 \rightarrow a & \text{if} \quad n = 1, \\ a_1 \rightarrow (a_2, \ldots, a_n; a) & \text{if} \quad n > 1. \end{cases}$$

A subset $F \subseteq A$ is an *implicative filter* or *deductive system* of A if $1 \in F$, and if $a, a \rightarrow b \in F$ then $b \in F$. The set of all implicative filters of a Hilbert algebra A is denoted by $\mathrm{Fi}(A)$. The implicative filter generated by a set X is $\langle X\rangle = \bigcap\{F \in \mathrm{Fi}(A) : X \subseteq F\}$. If $X = \{a\}$, then we write $\langle a\rangle = \{b \in A : a \leq b\}$. The implicative filter generated by a subset $X \subseteq A$ can be characterized as the set

$$\langle X\rangle = \{a \in A : \exists\{a_1, \ldots, a_n\} \subseteq X : (a_1, \ldots, a_n; a) = 1\}.$$

Let $F \in \mathrm{Fi}(A) - \{A\}$. We will say that F is *irreducible* if and only if for any $F_1, F_2 \in \mathrm{Fi}(A)$ such that $F = F_1 \cap F_2$, it follows that $F = F_1$ or $F = F_2$. The set of all irreducible implicative filters of a Hilbert algebra A is denoted by $X(A)$. Let us recall that an implicative filter F is irreducible iff for every $a, b \in A$ such that $a, b \notin F$ there exists $c \notin F$ such that $a, b \leq c$ (see [4], [11] or [24]). A subset I of A is called an *order-ideal* of A if $b \in I$ and $a \leq b$, then $a \in I$, and for each $a, b \in I$ there exists $c \in I$ such that $a \leq c$ and $b \leq c$. The set of all order-ideals of A will denoted by $\mathrm{Id}(A)$.

The following is a Hilbert algebra analogue of Birkhoff's Prime Filter Lemma and it is proved in [6]. We note that in [21] is used a similar theorem (see also [27]), but with the notion of a-maximal filter. It is not difficult to check that every a-maximal filter is irreducible, but the converse is not generally valid.

Theorem 2.2. *Let A be a Hilbert algebra. Let $F \in \mathrm{Fi}(A)$ and let $I \in \mathrm{Id}(A)$ such that $F \cap I = \emptyset$. Then, there exists $x \in X(A)$ such that $F \subseteq x$ and $x \cap I = \emptyset$.*

A *bounded Hilbert algebra* is a Hilbert algebra A with an element $0 \in A$ such that $0 \to a = 1$, for every $a \in A$. The notation $\neg a$ means $a \to 0$. The variety of bounded Hilbert algebras is denoted by Hil^0.

Lemma 2.3. *Let $A \in \mathrm{Hil}^0$. Then,*

1. *If $a \in x$, then $\neg a \notin x$, for every $x \in X(A)$.*
2. *If $\neg a \notin y$ then there exists $x \in X(A)$ such that $y \subseteq x$ and $a \in x$, for all $y \in X(A)$.*

Proof. (1) Suppose that $\neg a \in x$. So, $a \to 0 \in x$. As $a \in x$, we get that $0 \in x$, which is impossible because x is a proper implicative filter. (2) This is an immediate consequence of Theorem 2.2. □

For a partially ordered set $\langle X, \leq\rangle$ and $Y \subseteq X$, let

$$[Y) = \{x \in X : \exists y \in Y : y \leq x\}$$

and

$$(Y] = \{x \in X : \exists y \in Y : x \leq y\}.$$

If Y is the singleton $\{y\}$, then we write $[y)$ and $(y]$ instead of $[\{y\})$ and $(\{y\}]$, respectively. We call Y an *upset* (resp. *downset*) if $Y = [Y)$ (resp. $Y = (Y])$. The set of all upset subsets of X is denoted by $\mathrm{Up}(X)$. It is known that $\langle \mathrm{Up}(X), \Rightarrow_{\leq}, X\rangle$ is a Hilbert algebra where the implication $\Rightarrow_{\leq}$ is defined by

$$U \Rightarrow_{\leq} V = (U \cap V^c]^c = \{x : [x) \cap U \subseteq V\} \tag{1}$$

for $U, V \in \mathrm{Up}(X)$.

An *H-set* or *expanded Kripke frame* (in the terminology of Kirk in [21]) is a triple $\langle X, \leq, \mathcal{K}\rangle$ where $\langle X, \leq\rangle$ is a poset and $\emptyset \neq \mathcal{K} \subseteq \mathcal{P}(X)$. Every H-set defines a structure $H_{\mathcal{K}}(X)$ as follows:

$$H_{\mathcal{K}}(X) = \{U \in \mathcal{P}(X) : \exists W \in \mathcal{K} \text{ and } \exists V \subseteq W \quad (U = W \Rightarrow_{\leq} V)\}. \tag{2}$$

As is proved in [21] and [7] the triple $H_{\mathcal{K}}(X) = \langle H_{\mathcal{K}}(X), \Rightarrow_{\leq}, X\rangle$ is a Hilbert algebra and a subalgebra of $\langle \mathrm{Up}(X), \Rightarrow_{\leq}, X\rangle$. The algebra $H_{\mathcal{K}}(X)$ is called the *dual Hilbert algebra* of $\langle X, \leq, \mathcal{K}\rangle$.

Consider a pair $\langle X, \mathcal{K}\rangle$ where X is a set and $\emptyset \neq \mathcal{K} \subseteq \mathcal{P}(X)$. We define a relation $\leq_{\mathcal{K}} \subseteq X \times X$ by

$$x \leq_{\mathcal{K}} y \text{ iff } \forall W \in \mathcal{K}(x \notin W \text{ then } y \notin W). \tag{3}$$

It is easy to see that $\leq_{\mathcal{K}}$ is a reflexive and transitive relation. For each $Y \subseteq X$, let

$$\mathrm{sat}(Y) = \bigcap \{W : Y \subseteq W \ \& \ W \in \mathcal{K}\}$$

and

$$\mathrm{cl}(Y) = \bigcap \{X - W : Y \cap W = \emptyset \ \& \ W \in \mathcal{K}\}.$$

When $\mathcal{K}$ is a basis of a topology $\mathcal{T}$ defined on X, the relation $\leq_{\mathcal{K}}$ is the *specialization dual order* of X, $\mathrm{sat}(Y)$ is the *saturation* of Y, and $\mathrm{cl}(Y)$ is the *closure* of Y. We note that $\leq_{\mathcal{K}}$ can be defined in terms of the operator cl as follows: $x \leq_{\mathcal{K}} y$ iff $y \in \mathrm{cl}(\{x\}) = \mathrm{cl}(x)$. If X is T_0 then the relation $\leq_{\mathcal{K}}$ is a partial order. Moreover, if X is T_0 then $\mathrm{cl}(Y) = [Y)_{\leq_{\mathcal{K}}}$, $\mathrm{sat}(Y) = (Y]_{\leq_{\mathcal{K}}}$, and every open (resp. closed) subset is a downset (resp. upset) respect to $\leq_{\mathcal{K}}$.

Let X be a topological space. We recall that a subset $Y \subseteq X$ is *irreducible* provided for any closed subsets Y_1 and Y_2, if $Y = Y_1 \cup Y_2$ then $Y = Y_1$ or $Y = Y_2$. A topological space X is *sober* if, for every irreducible closed set Y, there exists a *unique* $x \in X$ such that $\mathrm{cl}(x) = Y$. Notice that a sober space is automatically T_0. A topological space $\langle X, \mathcal{T} \rangle$ with a base $\mathcal{K}$ we will denoted by $\langle X, \mathcal{T}_{\mathcal{K}} \rangle$ or simply by $\langle X, \mathcal{K} \rangle$. Recall that the relation $\leq_{\mathcal{K}}$ defined in (3) is an order when the space is T_0. From now on, for every sober topological space $\langle X, \mathcal{K} \rangle$ we shall write $\leq$ instead of $\leq_{\mathcal{K}}$.

Definition 2.4. [9] A *Hilbert space* or H-space is a topological space $\langle X, \mathcal{K} \rangle$ such that:

H1. $\mathcal{K}$ is a base of open and compact subsets for the topology $\mathcal{T}_{\mathcal{K}}$ on X,

H2. For every $A, B \in \mathcal{K}$, $\mathrm{sat}(A \cap B^c) \in \mathcal{K}$,

H3. $\langle X, \mathcal{K} \rangle$ is sober.

Let A be a Hilbert algebra. Let us consider the poset $\langle X(A), \subseteq \rangle$ and the mapping $\varphi : X(A) \to \mathrm{Up}\,(X(A))$ defined by

$$\varphi(a) = \{x \in X(A) : a \in x\}.$$

In [8] it was proved that the family $\mathcal{K}_A = \{\varphi\,(a)^c : a \in A\}$ is a basis for a topology $\mathcal{T}_{\mathcal{K}_A}$ and the pair $\langle X(A), \mathcal{K}_A \rangle$ is an H-space, called the *dual space of* A. If A is a bounded Hilbert algebra, then $\varphi(0) = \emptyset$. So, $X(A) = \varphi(0)^c \in \mathcal{K}_A$ and consequently the H-space $\langle X\,(A)\,, \mathcal{K}_A \rangle$ is compact.

If $\langle X, \mathcal{K} \rangle$ is an H-space, then for each $x \in X$, the set

$$\varepsilon\,(x) = \{U \in D\,(X) : x \in U\}$$

belongs to $X\,(D\,(X))$, where $D(X) = \{U : U^c \in \mathcal{K}\}$. Thus, the mapping $\varepsilon : X \to X\,(D\,(X))$ is well-defined and it is an homeomorphism between the topological spaces $\langle X, \mathcal{K} \rangle$ and $\left\langle X(D(X)), \mathcal{K}_{D(X)} \right\rangle$.

Let A and B be Hilbert algebras. A mapping $h : A \to B$ is a *semi-homomorphism* if $h(1) = 1$, and $h(a \to b) \leq h(a) \to h(b)$, for all $a, b \in A$. A mapping $h : A \to B$ is a *homomorphism* if h is a semi-homomorphism such that $h(a) \to h(b) \leq h(a \to b)$, for all $a, b \in A$. Note that a semi-homomophism is a monotone map.

Lemma 2.5. *Let A and B be Hilbert algebras. Let $h : A \to B$ be a semi-homomorphism. If $x \in X(A)$, then $(h(x^c)] \in \mathrm{Id}(B)$.*

Proof. Assume that $x \in X(A)$. Let $a, b \in (h(x^c)]$. Then there exist $c, d \notin x$ such that $a \leq h(c)$ and $b \leq h(d)$. Since x is irreducible, there exists $e \notin x$ such that $c, d \leq e$, and as h is monotonic, $a \leq h(e)$ and $b \leq h(e)$. So, $h(e) \in (h(x^c)]$, and thus $(h(x^c)]$ is an order-ideal. □

We denote by Hil$\mathcal{S}$ the category of H-algebras and semi-homomorphisms between Hilbert algebras. Similarly, we denote by Hil$\mathcal{H}$ the category of H-algebras and homomorphisms. Clearly, Hil$\mathcal{H}$ is a subcategory at Hil$\mathcal{S}$.

Definition 2.6. Let $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ be H-spaces. Let us consider a relation $R \subseteq X_1 \times X_2$. We say that R is an *H-relation* if $R^{-1}(U) \in \mathcal{K}_1$, for every $U \in \mathcal{K}_2$, and $R(x)$ is a closed subset of X_2, for all $x \in X_1$.

An H-relation $R \subseteq X_1 \times X_2$ is an *H-functional relation* if for each pair $(x, y) \in R$, there exists $z \in X_1$ such that $x \leq z$ and $R(z) = [y)$.

$\mathcal{SR}$ ($\mathcal{SRF}$) denote the category whose objects are H-spaces and whose morphisms are H-relations (H-functional relations). By Theorem 3.5 and Theorem 3.7 in [8] we have that the categories $\mathcal{SR}$ ($\mathcal{SRF}$) and Hil$\mathcal{S}$ (Hil$\mathcal{H}$) are dually equivalents.

3. $H\Box$-algebras: representation and duality

In this section we shall define the Hilbert algebras with a modal operator of necessity $\Box$.

Definition 3.1. A Hilbert algebra with a modal operator $\Box$, or $H\Box$-algebra for short, is a pair $A = \langle A, \Box\rangle$ where A is a Hilbert algebra and $\Box$ is a semi-homomorphism defined on A, i.e., $\Box 1 = 1$, and $\Box(a \to b) \leq \Box a \to \Box b$, for all $a, b \in A$.

We denote by $\mathrm{Hil}_\Box$ the variety of $H\Box$-algebras. The variety $\mathrm{Hil}_\Box$ correspond to the $\{\Box, \to\}$-reduct of the variety of Heyting algebras with a modal operator $\Box$ (see, for example [10]). Moreover, the variety of Tarski modal algebras introduced in [5] is a subvariety of $\mathrm{Hil}_\Box$.

Let A, $B \in \mathrm{Hil}_\Box$. A map $h : A \to B$ is a *$\Box$-semi-homomorphism* (*$\Box$-homomorphism*) if h is a semi-homomorphism (homomorphism) such that $h(\Box a) = \Box(h(a))$, for all $a \in A$. We denote by $\mathrm{Hil}_{\Box}\mathcal{S}$ the category of $H\Box$-algebras with $\Box$-semi-homomorphisms and by $\mathrm{Hil}_{\Box}\mathcal{H}$ the category of $H\Box$-algebras with $\Box$-homomorphisms.

Let X be a set and Q a binary relation defined on X. For each $U \in \mathcal{P}(X)$ consider the set

$$\Box_Q(U) = \{x \in X : Q(x) \subseteq U\}.$$

Example 3.2. [19] An *intuitionistic modal Kripke* frame is a relacional structure $\mathcal{F} = \langle X, \leq, Q\rangle$, where $\langle X, \leq\rangle$ is a poset, and Q is a binary relation defined on X such that $\leq \circ Q \subseteq Q \circ \leq$, where $\circ$ is the composition of relations. It is easy to see that $\langle \mathrm{Up}(X), \Rightarrow_{\leq}, \cap, \cup, \Box_Q, \emptyset, X\rangle$ is a Heyting algebra with a modal operator $\Box$. Thus, $\langle \mathrm{Up}(X), \Rightarrow_{\leq}, \Box_Q, X\rangle \in \mathrm{Hil}_{\Box}$.

Definition 3.3. A triple $\langle X, \mathcal{K}, Q\rangle$ is an *$H\Box$-frame* if $\langle X, \leq\rangle$ is a poset and $(\leq \circ Q) \subseteq (Q \circ \leq)$, where $\leq$ is $\leq_{\mathcal{K}}$.

An $H\Box$-frame $\langle X, \mathcal{K}, Q\rangle$ is a *general $H\Box$-frame* if:

1. $\mathrm{sat}(U \cap V^c) \in \mathcal{K}$, for every $U, V \in \mathcal{K}$.
2. $Q^{-1}(U) \in \mathcal{K}$, for every $U \in \mathcal{K}$.

Lemma 3.4. *If $\mathcal{F} = \langle X, \mathcal{K}, Q\rangle$ is a general $H\Box$-frame, then*

$$A(\mathcal{F}) = \langle \mathrm{Up}(X), \Rightarrow_{\leq}, \Box_Q, X\rangle \in \mathrm{Hil}_{\Box},$$

and $\langle D(X), \Box_Q\rangle$ is a subalgebra of $A(\mathcal{F})$.

Proof. As $\langle X, \leq\rangle$ is a poset, we have that $\langle \mathrm{Up}(X), \Rightarrow_{\leq}, X\rangle$ is a Hilbert algebra. We note that $\Box_Q(U) \in \mathrm{Up}(X)$, for every $U \in \mathrm{Up}(X)$, because $(\leq \circ Q) \subseteq (Q \circ \leq)$. Moreover, as $\Box_Q(U) = Q^{-1}(U^c)^c$ we get that $\Box_Q(U) \in D(X)$, because $Q^{-1}(U^c) \in \mathcal{K}$ for every $U \in D(X)$. Finally, it is immediate to see that $\langle D(X), \Rightarrow_{\leq_{\mathcal{K}}}, X\rangle$ is a subalgebra of the Hilbert algebra $\langle \mathrm{Up}(X), \Rightarrow_{\leq_{\mathcal{K}}}, X\rangle$. $\Box$

Let $A \in \mathrm{Hil}_{\Box}$. For each $n \geq 0$, $n \in \mathbb{N}$, we define inductively the formula $\Box^n a$ as $\Box^0 a = a$ and $\Box^{n+1} a = \Box(\Box^n a)$. Let S be a subset of A. We define the following sets:

$$\Box(S) = \{\Box a \in A : a \in S\} \text{ and } \Box^{-1}(S) = \{a \in A : \Box a \in S\}.$$

We note that $\Box^{-1}(F) \in \mathrm{Fi}(A)$, when $F \in \mathrm{Fi}(A)$. We note also that by Lemma 2.5 $(\Box(x^c)]$ is an order-ideal, when $x \in X(A)$.

Lemma 3.5. *Let $A \in \mathrm{Hil}_{\Box}$. Let $F \in \mathrm{Fi}(A)$ and $a \in A$. Then $\Box a \notin F$ iff there exists $x \in X(A)$ such that $\Box^{-1}(F) \subseteq x$ and $a \notin x$.*

Proof. The proof follows taking into account that $\Box^{-1}(F)$ is an implicative filter and Theorem 2.2. $\Box$

Let A be an $H\Box$-algebra. By the results given in [8], the binary relation $Q_A \subseteq X(A) \times X(A)$ given by

$$(x, y) \in Q_A \text{ iff } \Box^{-1}(x) \subseteq y,$$

for $x, y \in X(A)$, is the H-relation associated with the modal operator $\Box$. So, $Q_A^{-1}(U) \in \mathcal{K}_A$, for every $U \in \mathcal{K}_A$. It is easy to see that Q_A satisfies the condition $Q_A = (\subseteq \circ Q_A) = (Q_A \circ \subseteq)$. Moreover, by Proposition 2.1 in [8] we have that if $U, V \in \mathcal{K}_A$, then $\mathrm{sat}(U \cap V^c) \in \mathcal{K}_A$. Thus, the triple

$$\mathcal{F}(A) = \langle X(A), \mathcal{K}_A, Q_A \rangle,$$

is a general $H\Box$-frame.

Now we shall define the $H\Box$-spaces, and we will see that its structures are a particular class of general $H\Box$-frames.

Definition 3.6. A triple $\langle X, \mathcal{K}, Q \rangle$ is an *$H\Box$-space* if $\langle X, \mathcal{K} \rangle$ is an H-space and $Q \subseteq X \times X$ is an H-relation.

As Q is an H-relation in every $H\Box$-space $\langle X, \mathcal{K}, Q \rangle$, by Teorem 3.1.(1) in [8] we get that $(\leq \circ Q) = Q = (Q \circ \leq)$ is valid in any $H\Box$-space. Consequently, we have the following result.

Lemma 3.7. *Every $H\Box$-space is a general $H\Box$-frame.*

Thus, if $\langle X, \mathcal{K}, Q \rangle$ is an $H\Box$-space, then $\langle D(X), \Box_Q \rangle$ is an $H\Box$-algebra.

Theorem 3.8 (of Representation)**.** *For each $H\Box$-algebra $\langle A, \Box \rangle$ there exists an $H\Box$-space $\langle X, \mathcal{K}, Q \rangle$ such that $\langle A, \Box \rangle$ is isomorphic to $\langle D(X), \Box_Q \rangle$.*

Proof. Since $\langle X(A), \mathcal{K}_A \rangle$ is an H-space and Q_A is an H-relation, we have that $\langle X(A), \mathcal{K}_A, Q_A \rangle$ is an $H\Box$-space. By Lemma 3.5, we have that $\varphi(\Box a) = \Box_{Q_A}(\varphi(a))$, for each $a \in A$. So, $\langle D(X(A)), \Box_{Q_A} \rangle$ is an $H\Box$-algebra. By Theorem 2.1 in [8] we get that φ is a Hilbert isomorphism. Thus, $\langle A, \Box \rangle$ is isomorphic to $\langle D(X(A)), \Box_{Q_A} \rangle$. $\Box$

Definition 3.9. Let $\langle X_1, \mathcal{K}_1, Q_1\rangle$ and $\langle X_2, \mathcal{K}_2, Q_2\rangle$ be $H\Box$-spaces and $R \subseteq X_1 \times X_2$ be an H-relation. We say that R is an $H\Box$*-relation* if R commutes with Q, i.e., $Q_1 \circ R = R \circ Q_2$.

If $R \subseteq X_1 \times X_2$ is an H-functional relation such that R commutes with Q, then R is an $H\Box$*-functional relation.*

$\mathcal{M}_\Box\mathcal{SR}$ denote the category of $H\Box$-spaces and $H\Box$-relations. We will prove that this category is dually equivalent to $\mathrm{Hil}_\Box\mathcal{S}$.

Let $\langle X, \mathcal{K}\rangle$ an H-space and consider the map $\varepsilon : X \to X\,(D(X))$ defined by $\varepsilon\,(x) = \{U \in D\,(X) : x \in U\}$. By Corollary 3.1 in [8] we get that the relation $\varepsilon^* \subseteq X \times X\,(D(X))$ given by

$$(x, P) \in \varepsilon^* \text{ iff } \varepsilon(x) \subseteq P$$

is an H-relation. Now, we will prove that ε^* is a morphism of $H\Box$-spaces.

Theorem 3.10. *Let $\langle X, \mathcal{K}, Q\rangle$ an $H\Box$-space. Then, the mapping ε is an homeomorphism between the $H\Box$-spaces $\langle X, \mathcal{K}, Q\rangle$ and $\langle X(D(X)), \mathcal{K}_{D(X)}, Q_{D(X)}\rangle$ such that*

$$(x, y) \in Q \quad \text{iff} \quad (\varepsilon(x), \varepsilon(y)) \in Q_{D(X)},$$

where $Q_{D(X)}$ is the $H\Box$-relation associated with the modal operator $\Box_Q$. Moreover, the relation ε^* is a morphism of $H\Box$-spaces.

Proof. As $\langle X, \mathcal{K}, Q\rangle$ is an $H\Box$-space, $\langle D\,(X)\,, \Box_Q\rangle$ is an $H\Box$-algebra and by Theorem 3.8, the triple $\left\langle X\,(D\,(X))\,, \mathcal{K}_{D(X)}, Q_{D(X)}\right\rangle$ is an $H\Box$-space where $(F, P) \in Q_{D(X)}$ iff $\Box_Q^{-1}(F) \subseteq P$, for all $F, P \in X(D(X))$. By Theorem 2.2 in [8] we get that ε is an homeomorphism between the H-spaces $\langle X, \mathcal{K}\rangle$ and $\left\langle X\,(D\,(X))\,, \mathcal{K}_{D(X)}\right\rangle$, being $\mathcal{K}_{D(X)} = \{\varphi(U)^c : U \in D(X)\}$.

Let $(x, y) \in Q$. We prove that $(\varepsilon(x), \varepsilon(y)) \in Q_{D(X)}$, i.e., $\Box_Q^{-1}(\varepsilon(x)) \subseteq \varepsilon(y)$. Let $U \in D(X)$ such that $U \in \Box_Q^{-1}(\varepsilon(x))$. So, $Q(x) \subseteq U$ and as $y \in Q(x)$, we get that $y \in U$. This is, $U \in \varepsilon(y)$. Now, assume that $\Box_Q^{-1}(\varepsilon(x)) \subseteq \varepsilon(y)$ and suppose that $(x, y) \notin Q$. As $Q(x)$ is a closed subset of $\langle X, \mathcal{K}\rangle$, there exists $U \in D(X)$ such that $Q(x) \subseteq U$ and $y \notin U$. This is, $U \in \Box_Q^{-1}(\varepsilon(x))$ and $U \notin \varepsilon(y)$, which contradicts the assumption.

Now, we will prove that $Q \circ \varepsilon^* = \varepsilon^* \circ Q_{D(X)}$. Let $x \in X$ and $P \in X\,(D(X))$ such that $(x, P) \in Q \circ \varepsilon^*$. So, there exists $y \in X$ such that $(x, y) \in Q$ and $(y, P) \in \varepsilon^*$. This is, $\varepsilon(y) \subseteq P$. As $(x, y) \in Q$, we have

$(\varepsilon(x), \varepsilon(y)) \in Q_{D(X)}$, i.e., $\Box_Q^{-1}(\varepsilon(x)) \subseteq \varepsilon(y) \subseteq P$. Thus, $(\varepsilon(x), P) \in Q_{D(X)}$. It is clear that $(x, \varepsilon(x)) \in \varepsilon^*$. So, $(x, P) \in \varepsilon^* \circ Q_{D(X)}$. Thus, $Q \circ \varepsilon^* \subseteq \varepsilon^* \circ Q_{D(X)}$. Assume that $(x, P) \in \varepsilon^* \circ Q_{D(X)}$. So, there exists $F \in X\ (D(X))$ such that $\varepsilon(x) \subseteq F$ and $\Box_Q^{-1}(F) \subseteq P$. As ε is onto, there exists $f, p \in X$ such that $F = \varepsilon(f)$ and $P = \varepsilon(p)$. So, $\Box_Q^{-1}(\varepsilon(x)) \subseteq \Box_Q^{-1}(\varepsilon(f)) \subseteq \varepsilon(p)$. Then, $(\varepsilon(x), \varepsilon(p)) \in Q_{D(X)}$ and consequently, $(x, p) \in Q$. It is clear that $(p, P) \in \varepsilon^*$. So, $(x, P) \in Q \circ \varepsilon^*$. □

In [8] it was proved that if $\langle X_1, \mathcal{K}_1\rangle$ and $\langle X_2, \mathcal{K}_2\rangle$ are H-spaces and $R \subseteq X_1 \times X_2$ is an H-relation then the mapping $h_R : D(X_2) \to D(X_1)$ defined by

$$h_R(U) = \{x \in X_1 \mid R(x) \subseteq U\}$$

is a semi-homomorphism.

Theorem 3.11. *Let $\langle X_1, \mathcal{K}_1, Q_1\rangle$ and $\langle X_2, \mathcal{K}_2, Q_2\rangle$ be $H\Box$-spaces and $R \subseteq X_1 \times X_2$ be an $H\Box$-relation. Then, h_R is a morphism of* $\mathrm{Hil}_\Box\mathcal{S}$.

Proof. We will prove that $h_R(\Box_{Q_2}(U)) = \Box_{Q_1}(h_R(U))$, for each $U \in D(X_2)$. Let $x \in X_1$. Then

$$\begin{array}{llllll} x \in h_R(\Box_{Q_2}(U)) & \text{iff} & R(x) \subseteq \Box_{Q_2}(U) & \text{iff} & Q_2(R(x)) \subseteq U \\ & \text{iff} & R(Q_1(x)) \subseteq U & \text{iff} & \forall z \in Q_1(x)(R(z) \subseteq U) \\ & \text{iff} & Q_1(x) \subseteq h_R(U) & \text{iff} & x \in \Box_{Q_1}(h_R(U)). \end{array}$$

□

By the above Theorem and Theorem 3.7 in [8], we have the following result.

Corollary 3.12. *Let $\langle X_1, \mathcal{K}_1, Q_1\rangle$ and $\langle X_2, \mathcal{K}_2, Q_2\rangle$ be $H\Box$-spaces and $R \subseteq X_1 \times X_2$ be an $H\Box$-functional relation. Then, h_R is a morphism of* $\mathrm{Hil}_\Box\mathcal{H}$.

Let A, B be Hilbert algebras and $h : A \to B$ be a semi-homomorphism. In [8] it was proved that the relation $R_h \subseteq X(B) \times X(A)$ defined by

$$(x, y) \in R_h \ \text{ iff } \ h^{-1}(x) \subseteq y$$

is an H-relation. Now, we will study R_h when h is a semi-homomorphism defined between $H\Box$-algebras that commutes with $\Box$.

Theorem 3.13. *Let $A, B \in \mathrm{Hil}_\Box$ and let $h : A \to B$ be a $\Box$-semi-homomorphism. Then, R_h is a morphism of $\mathcal{M}_\Box\mathcal{SR}$.*

Proof. If we prove that $R_h \circ Q_A = Q_B \circ R_h$, the assertion follows. Let $x \in X(B)$ and $y \in X(A)$ such that $(x, y) \in R_h \circ Q_A$. So, there exists $z \in X(A)$ such that $z \in R_h(x)$ and $(z, y) \in Q_A$, i.e., $h^{-1}(x) \subseteq z$ and $\Box^{-1}(z) \subseteq y$. Consider the implicative filter $\Box^{-1}(x)$ and the order-ideal $(h(y^c)]$ of B. Suppose that there exists $a \in \Box^{-1}(x) \cap (h(y^c)]$. So, $\Box a \in x$ and there exists $b \in y^c$ such that $a \leq h(b)$. As $\Box a \leq \Box(h(b)) = h(\Box b)$, we get that $h(\Box b) \in x$. Thus, $\Box b \in z$ and so, $b \in y$, which is a contradiction. Thus, $\Box^{-1}(x) \cap (h(y^c)] = \emptyset$. So, there exists $w \in X(B)$ such that $\Box^{-1}(x) \subseteq w$ and $(h(y^c)] \cap w = \emptyset$. This is, there exists $w \in X(B)$ such that $w \in Q_B(x)$ and $h^{-1}(w) \subseteq y$, i.e., $(w, y) \in R_h$. Therefore, $y \in R_h(Q_B(x))$. Thus, $R_h \circ Q_A \subseteq Q_B \circ R_h$. The proof of the other inclusion is similar. $\Box$

By Theorem 3.13 and Theorem 3.7 in [8] we have the following result.

Corollary 3.14. *Let $A, B \in \mathrm{Hil}_\Box$ and let $h : A \to B$ be a $\Box$-homomorphism. Then R_h is an $H\Box$-functional relation.*

From Theorem 3.11, we conclude that the functor $\mathbb{D} : \mathcal{M}_\Box\mathcal{SR} \to \mathrm{Hil}_\Box\mathcal{S}$ defined by

$$\begin{array}{ll} \mathbb{D}(X) = \langle D(X), \Box_Q \rangle & \text{if } \langle X, \mathcal{K}, Q \rangle \text{ is an } H\Box\text{-space}, \\ \mathbb{D}(R) = h_R & \text{if } R \text{ is an } H\Box\text{-relation.} \end{array}$$

is a contravariant functor. By Remark 3.1 in [8], Theorem 3.8 and Theorem 3.13, we conclude that the functor $\mathbb{X} : \mathrm{Hil}_\Box\mathcal{S} \to \mathcal{M}_\Box\mathcal{SR}$ defined by

$$\begin{array}{ll} \mathbb{X}(A) = \langle X(A), \mathcal{K}_A, Q_A \rangle & \text{if } A \text{ is an } H\Box\text{-algebra}, \\ \mathbb{X}(h) = R_h & \text{if } h \text{ is a } \Box\text{-semi-homomorphism} \end{array}$$

is a contravariant functor. From the Lemmas 3.4 and 3.5 in [8] and Theorems 3.8 and 3.10, we give the following result.

Theorem 3.15. *The categories $\mathrm{Hil}_\Box\mathcal{S}$ and $\mathcal{M}_\Box\mathcal{SR}$ are dually equivalent.*

Corollary 3.16. *The category $\mathrm{Hil}_\Box\mathcal{H}$ is dually isomorphic to the category of $H\Box$-spaces with $H\Box$-functional relations.*

4. Some subvarieties of $H\Box$-algebras

The variety of $H\Box$-algebras generated by a finite set of identities Γ will be denoted by $\mathrm{Hil}_\Box + \{\Gamma\}$. We shall consider some particular varieties of $H\Box$-algebras. These varieties are the algebraic counterpart of extensions of the implicative fragments of the intuitionistic modal logic $\mathbf{IntK}_\Box$. Let us consider the following identities:

$\mathbf{S}$	$a \to \Box a \approx 1,$
$\mathbf{S}_n$	$a \to \Box^n a \approx 1,$
$\mathbf{T}$	$\Box a \to a \approx 1,$
$\mathbf{4}$	$\Box a \to \Box^2 a \approx 1,$
$\mathbf{wD}$	$\Box^2 a \to \Box a \approx 1,$
$\mathbf{5}$	$(\Box a \to \Box b) \to \Box(\Box a \to \Box b) \approx 1,$
$\mathbf{6}$	$\Box^2 a \to \Box a \approx 1.$

Remark 4.1. It is not hard to prove that $\mathrm{Hil}_\Box + \{\mathbf{5}\}$ and $\mathrm{Hil}_\Box + \{\mathbf{S}\}$ are subvarieties of $\mathrm{Hil}_\Box + \{\mathbf{4}\}$.

Following the standard notation, we shall identify two important subvarieties of $\mathrm{Hil}_\Box$:

$$\begin{aligned} \mathrm{Hil}_\Box\mathbf{S4} &= \mathrm{Hil}_\Box + \{\mathbf{T}, \mathbf{4}\}, \\ \mathrm{Hil}_\Box\mathbf{S5} &= \mathrm{Hil}_\Box + \{\mathbf{T}, \mathbf{5}\}. \end{aligned}$$

It is clear that $\mathrm{Hil}_\Box\mathbf{S5}$ is subvariety of $\mathrm{Hil}_\Box\mathbf{S4}$. The variety $\mathrm{Hil}_\Box\mathbf{S4}$ is a generalization of the topological o closure Boolean algebras, and the variety $\mathrm{Hil}_\Box\mathbf{S5}$ is a generalization of the monadic Boolean algebras. Similar to the proven in [5], each one of the previous identities are characterized by means of first-order conditions.

Let Q be a binary relation defined on a set X. For each $n \geq 0$ we define inductively the relation Q^n as follows: $(x, y) \in Q^0$ iff $x = y$, and $(x, y) \in Q^{n+1} = Q^n \circ Q$, where $\circ$ is the composition of relations. Also we define the binary relation $Q^* = \bigcup \{Q^n : n \geq 0\}$.

The next result is a generalization of Lemma 3.5 applied to irreducible implicative filters.

Lemma 4.2. *Let $A \in \mathrm{Hil}_\Box$ and let $\langle X, \mathcal{K}, Q\rangle$ be its dual space. Let $x \in X$ and $a \in A$. For each $n \in \mathbb{N}$, $\Box^n a \notin x$ iff there exists $y \in X$ such that $(x, y) \in Q^n$ and $a \notin y$.*

Proof. The proof is by induction on n. It is inmediatly for $n = 0$. Assume that $\Box^n a \notin x$ implies that there exists $y \in X$ such that $(x, y) \in Q^n$ and $a \notin y$. Suppose that $\Box^{n+1} a \notin x$. This is, $\Box(\Box^n a) \notin x$. By Lemma 3.5, there exists $y \in X$ such that $\Box^{-1}(x) \subseteq y$ and $\Box^n a \notin y$. By assumption, there exists $z \in X$ such that $(y, z) \in Q^n$ and $a \notin z$. Since $(x, y) \in Q$ and $(y, z) \in Q^n$, we get that $(x, z) \in Q^{n+1}$.

Consider that if there exists $y \in X$ such that $(x, y) \in Q^n$ and $a \notin y$, then $\Box^n a \notin x$. Suppose that $(x, y) \in Q^{n+1}$ and $a \notin x$. So, there exists $z \in X$ such that $(x, z) \in Q^n$ and $(z, y) \in Q$. Therefore, $\Box^{-1}(z) \subseteq y$ and as $a \notin y$, we have that $\Box a \notin z$. Thus, $(x, z) \in Q^n$ and $\Box a \notin z$. By assumption, $\Box^{n+1} a \notin x$. $\Box$

Let $\langle X, \mathcal{K}, Q \rangle$ be an $H\Box$-space. Following the notation used in [19], we denote by Φ and Φ' the next first-order conditions:

$$\Phi \quad \Leftrightarrow \quad \forall x \forall y \left[xQy \wedge yQz \Rightarrow \exists w(x \leq w \wedge wQz \wedge \forall v(wQv \Rightarrow yQv)) \right].$$
$$\Phi' \quad \Leftrightarrow \quad \forall x \forall y \left[xQy \wedge yQz \Rightarrow \exists w(x \leq w \wedge wQz \wedge yQw) \right].$$

Remark 4.3. Let $\langle X, \mathcal{K}, Q \rangle$ be an $H\Box$-space. Note that Φ' (or Φ) implies the transitivity of Q. In fact. Let $x, y, z \in X$ such that xQy and yQz. By Φ', there exists $w \in X$ such that $x \leq w$, wQz and yQw. By Lemma 3.7, $(x, z) \in Q$. This result us allows to prove that if Q is reflexive then, Φ' and Φ are equivalent. For this is enough to show that $\forall v(wQv \Rightarrow yQv) \Leftrightarrow yQw$. From left to right we use wQw. For the other direction, suppose that yQw and wQv, for every $v \in X$ and use that Φ' implies the transitivity of Q.

Theorem 4.4. *Let $A \in \mathrm{Hil}_\Box$ and let $\langle X, \mathcal{K}, Q \rangle$ be its dual space. Then:*

1. $A \vDash a \to \Box a \approx 1$ *iff* $\forall x \forall y\, (xQy \Rightarrow x \subseteq y)$.
2. $A \vDash a \to \Box^n a \approx 1$ *iff* $\forall x \forall y (xQ^n y \Rightarrow x \subseteq y)$, *with* $n \in \mathbb{N}$.
3. $A \vDash \Box a \to a \approx 1$ *iff* Q *is reflexive.*
4. $A \vDash \Box a \to \Box^2 a \approx 1$ *iff* Q *is transitive.*
5. $A \vDash \Box^2 a \to \Box a \approx 1$ *iff* Q *is weakly dense, i.e.,* $\forall x \forall y (xQy \Rightarrow \exists z (xQz \wedge zQy))$.
6. $A \vDash \Box(\Box a \to a) \approx 1$ *iff* $\forall x \forall y (xQy \Rightarrow yQy)$.
7. $A \vDash (\Box a \to \Box b) \to \Box(\Box a \to \Box b) \approx 1$ *iff* $\langle X, \mathcal{K}, Q \rangle$ *satisfies* Φ.

Proof. We will prove only the assertions (2), (5) and (7). The other proofs are analogous.

(2) Let $n \in \mathbb{N}$. Suppose that there exist $x, y \in X$ such that $(x, y) \in Q^n$ and $x \nsubseteq y$. Hence, there is an element $a \in x$ such that $a \notin y$. As $(x, y) \in Q^n$ and $a \notin y$, by Lemma 4.2, $\Box^n a \notin x$. Since $a \leq \Box^n a$, we have that $a \notin x$, which is a contradiction. Reciprocally, if there exists $a \in A$ such that $a \nleq \Box^n a$ then, there exists $x \in X$ such that $a \in x$ and $\Box^n a \notin x$. By Lemma 4.2, we get an irreducible implicative filter $y \in X$ such that $(x, y) \in Q^n$ and $a \notin y$. By assumption, $x \subseteq y$ and so, $a \notin x$, which is impossible.

(5) Assume that $\Box^2 a \leq \Box a$ for all $a \in A$ and let $(x, y) \in Q$. Consider the implicative filter $\Box^{-1}(x)$ and the order-ideal $(\Box(y^c)]$. Suppose that there exists $a \in \Box^{-1}(x) \cap (\Box(y^c)]$. So, $\Box a \in x$ and there exists $p \in y^c$ such that $a \leq \Box p$. Thus, $\Box a \leq \Box^2 p \leq \Box p$ and consequently, $\Box p \in x$. So, $p \in \Box^{-1}(x)$. As $(x, y) \in Q$, we have that $p \in y$, which is impossible. So, $\Box^{-1}(x) \cap (\Box(y^c)] = \emptyset$. Thus, by Theorem 2.2, there exists $z \in X$ such that $\Box^{-1}(x) \subseteq z$ and $z \cap (\Box(y^c)] = \emptyset$. This is, $z \subseteq \Box(y^c)^c$ and so, $\Box^{-1}(z) \subseteq y$. Thus, we have that there exists $z \in X$ such that $(x, z) \in Q$ and$(z, y) \in Q$. Reciprocally. Suppose that there exists $a \in A$ such that $\Box^2 a \nleq \Box a$. So, there exists $x \in X$ such that $\Box^2 a \in x$ and $\Box a \notin x$. By Lemma 4.2, there exists $y \in X$ such that $(x, y) \in Q$ and $a \notin y$. By assumption, $(x, y) \in Q^2$ and as $a \notin y$, we get that $\Box^2 a \notin x$, which is a contradiction.

(7) Consider that $(\Box a \to \Box b) \leq \Box(\Box a \to \Box b)$, for every $a, b \in A$. Let $(x, y) \in Q$ and $(y, z) \in Q$. Note that the implicative filter $\langle x \cup \Box(\Box^{-1}(y)) \rangle$ and the order-ideal $(\Box(z^c)]$ are disjoint. Indeed, suppose that there exists $a \in A$ such that $a \in \langle x \cup \Box(\Box^{-1}(y)) \rangle$ and $a \in (\Box(z^c)]$. Thus, by the characterization of implicative filter generated by a set given on page 50, there exist $b \in x$, $c \in \Box^{-1}(y)$, and $d \notin z$ such that $b \to (\Box c \to a) = 1$ and $a \leq d$. So, we have that $1 = b \to (\Box c \to a) \leq b \to (\Box c \to d)$. Then, $b \to (\Box c \to \Box d) = 1 \in x$. Thus, $\Box c \to \Box d \in x$. As $\Box c \to \Box d \leq \Box(\Box c \to \Box d)$, we get that $\Box(\Box c \to \Box d) \in x$. So, $\Box c \to \Box b \in \Box^{-1}(x)$ and by assumption, $\Box c \to \Box d \in y$. As $\Box c \in y$, we get that $\Box d \in y$ and so, $d \in z$, which is a contradiction. Thus, by Theorem 2.2 we can affirm that there exists $w \in X$ such that $x \subseteq w$, $\Box(\Box^{-1}(y)) \subseteq w$ and $\Box(z^c) \cap w = \emptyset$. Hence, $\Box^{-1}(y) \subseteq \Box^{-1}(w)$ and $\Box^{-1}(w) \subseteq z$. For every $v \in X$ such that $(w, v) \in Q$, we get that $\Box^{-1}(y) \subseteq \Box^{-1}(w) \subseteq v$. So, $(y, v) \in Q$. We have proved that $\langle X, \mathcal{K}, Q \rangle$ satisfies the condition Φ.

Conversely. Suppose that there exist $a, b \in A$ such that $\Box a \to \Box b \nleq$

$\Box(\Box a \to \Box b)$. So, there exists $x \in X$ such that $\Box a \to \Box b \in x$ and $\Box(\Box a \to \Box b) \notin x$. Then, there exists $y \in X$ such that $\Box^{-1}(x) \subseteq y$ and $\Box a \to \Box b \notin y$. By consequence of Theorem 2.2, there exists $z \in X$ such that $y \subseteq z$, $\Box a \in z$ and $\Box b \notin z$. So, there exists $w \in X$ such that $\Box^{-1}(z) \subseteq w$ and $b \notin w$. Thus, $(x,z) \in Q$ and $(z,w) \in Q$. By assumption, there exists $v \in X$ such that $x \subseteq v$, $(v,w) \in Q$ and for all $u \in X$ such that $(v,u) \in Q$, we can affirm that $(z,u) \in Q$. Since $\Box a \to \Box b \in x$, we have that $\Box a \to \Box b \in v$. On the other hand, $b \notin w$ and so, $\Box b \notin v$. Thus, $\Box a \notin v$ and consequently, there exists $u \in X$ such that $(v,u) \in Q$ and $a \notin u$. Hence, $(z,u) \in Q$, and so, $\Box a \notin z$, which is impossible. $\Box$

We shall say that an $H\Box$-algebra $\langle A, \Box \rangle$ is *bounded* if the Hilbert algebra A is bounded. The variety of bounded $H\Box$-algebras is denoted by $\mathrm{Hil}^0_\Box$.

Theorem 4.5. *Let $A \in \mathrm{Hil}^0_\Box$ and let $\langle X, \mathcal{K}, Q \rangle$ be its dual space. Then,*

1. *$A \vDash \Box 0 \to 0 \approx 1$ iff Q is serial, i.e., $\forall x \exists y (xQy)$.*
2. *If Q is reflexive and transitive, we have that $A \vDash \neg\Box a \to \Box\neg\Box a \approx 1$ iff $Q \subseteq \left(\subseteq \circ Q^{-1}\right)$.*

Proof. (1) Suppose that $\Box 0 = 0$. Since $0 \notin x$ for all $x \in X$, we get that $0 \notin \Box^{-1}(x)$. Thus, for each $x \in X$ there exists $y \in X$ such that $\Box^{-1}(x) \subseteq y$ and $0 \notin y$. So, Q is serial. Conversely. Suppose that $\Box 0 \nleq 0$. There is $x \in X$ such that $\Box 0 \in x$ and $0 \notin x$. Hence, $0 \in \Box^{-1}(x)$ and by assumption, there exists $y \in X$ such that $\Box^{-1}(x) \subseteq y$. Thus, $0 \in y$, which is impossible.

(2) Let Q be reflexive and transitive. Assume that $\neg\Box a \leq \Box\neg\Box a$ for all $a \in A$ and let $(x,y) \in Q$. Suppose that $0 \in \left\langle x \cup \Box(\Box^{-1}(y))\right\rangle$. So, there exist $a \in x$ and $b \in \Box^{-1}(y)$ such that $a \to (\Box b \to 0) = 1$, this is, $a \leq \neg\Box b$. Thus, $\neg\Box b \in x$ and so, $\Box\neg\Box b \in x$. Thus, $\neg\Box b \in \Box^{-1}(x)$ and consequently, $\Box b \to 0 \in y$. As $\Box b \in y$, then $0 \in y$, which is impossible. So, there exists $z \in X$ such that $\left\langle x \cup \Box(\Box^{-1}(y))\right\rangle \subseteq z$ and $0 \notin z$. Hence, $x \subseteq z$ and $\Box(\Box^{-1}(y)) \subseteq z$. So, $\Box^{-1}(y) \subseteq \Box^{-1}(z)$. As Q is reflexive, $\Box^{-1}(z) \subseteq z$ and so, $(y,z) \in Q$. Thus, $(x,y) \in \left(\subseteq \circ Q^{-1}\right)$.

Reciprocally. Assume that there is an element $a \in A$ such that $\neg\Box a \nleq \Box\neg\Box a$. So, there exist $x, y \in X$ such that $\neg\Box a \in x$, $\Box\neg\Box a \notin x$, $\Box^{-1}(x) \subseteq y$ and $\neg\Box a \notin y$. By Lemma 2.3, we have an irreducible implicative filter z such that $y \subseteq z$ and $\Box a \in z$. Thus, $(x,z) \in Q$ and $\Box a \in z$. By assumption, there exists $w \in X$ such that $x \subseteq w$ and $(z,w) \in Q$. As $\neg\Box a \in x$, we have $\neg\Box a \in w$. So, $\Box a \notin w$, implying that $\Box^2 a \notin z$. As Q is transitive, by Theorem 4.4, we have that $\Box a \leq \Box^2 a$. So, $\Box a \notin z$, which is impossible. □

We shall identify some subvarieties of $\mathrm{Hil}^0_\Box$:

$$\begin{aligned} \mathrm{Hil}^0_\Box\mathbf{S5} &= \mathrm{Hil}^0_\Box + \{\mathbf{T}, \mathbf{5}\}, \\ \mathrm{Hil}^0_\Box\mathbf{S5.1} &= \mathrm{Hil}^0_\Box + \{\mathbf{T}, \mathbf{4}, \neg\Box a \to \Box\neg\Box a \approx 1\}, \\ \mathrm{Hil}^w_\Box\mathbf{S5} &= \mathrm{Hil}^0_\Box + \{\mathbf{5}, \Box 0 \to 0 \approx 1\}. \end{aligned}$$

Note that $\mathrm{Hil}^0_\Box\mathbf{S5}$ is subvariety of $\mathrm{Hil}^0_\Box\mathbf{S5.1}$ and $\mathrm{Hil}^w_\Box\mathbf{S5}$. Indeed. If $A \in \mathrm{Hil}^0_\Box\mathbf{S5}$, we have that $\Box a \to a \approx 1$, in particular, $\Box 0 \to 0 \approx 1$. Thus, $A \in \mathrm{Hil}^w_\Box\mathbf{S5}$. Moreover, by Remark 4.1, $\Box a \to \Box^2 a \approx 1$ and as for all $a \in A$, $1 = (\Box a \to 0) \to \Box(\Box a \to 0) = \neg\Box a \to \Box\neg\Box a$, we get that $A \in \mathrm{Hil}^0_\Box\mathbf{S5.1}$.

It is clear that $\mathrm{Hil}^0_\Box\mathbf{S5.1}$ is subvariety of $\mathrm{Hil}^0_\Box\mathbf{S4}$ and consequently, $\mathrm{Hil}^0_\Box\mathbf{S5}$ is subvariety of $\mathrm{Hil}^0_\Box\mathbf{S4}$.

Corollary 4.6. *Let $A \in \mathrm{Hil}^0_\Box$ and $\langle X, \mathcal{K}, Q\rangle$ be its dual space. Then, $A \in \mathrm{Hil}^0_\Box\mathbf{S5.1}$ iff Q is reflexive, transitive and $Q \subseteq \big(\subseteq \circ Q^{-1}\big)$.*

Proof. By Theorem 4.4 and previous Theorem. □

5. Implicational modal logics

In this section we shall define the $\{\to, \Box\}$-fragment of the intuitionistic normal modal logic $\mathbf{IntK}_\Box$ and some of its extensions. Let $\mathcal{L}$ be the propositional modal language with an infinite set of propositional variables Var, a propositional constant $\top$, the connective $\to$, and the unary operator $\Box$. The set of all formulas of $\mathcal{L}$, we denote by Fm.

The logic $\mathbf{IntK}^{\to}_\Box$ is a logic in the language $\mathcal{L}_\Box$ characterized by the following list of axioms and rules:

1. $\phi \to (\psi \to \phi)$,
2. $(\phi \to (\psi \to \alpha)) \to ((\phi \to \psi) \to ((\phi \to \alpha))$,
3. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$,

(MP) $\dfrac{\phi, \phi \to \psi}{\psi}$, (N) $\dfrac{\phi \to \psi}{\Box\phi \to \Box\psi}$.

It is clear that $\mathbf{IntK}_{\Box}^{\to}$ is the $\{\Box, \to\}$-fragment of intuitionistic modal logic $\mathbf{IntK}_{\Box}$. An *implicational modal logic* $\mathcal{I}_{\Box}$ is any extension of $\mathbf{IntK}_{\Box}^{\to}$.

Let $\mathcal{F} = \langle X, \mathcal{K}, Q\rangle$ be an $H\Box$-frame or a general $H\Box$-frame (see Definition 3.3). A *valuation* on $\mathcal{F}$ is a function $V : Var \to \mathrm{Up}(X)$ ($V : Var \to D(X)$) on the $H\Box$-frame (general $H\Box$-frame) $\mathcal{F}$. As is usual, V is extended recursively to algebra of all formulas Fm by means of the clauses

1. $V(\top) = X$,
2. $V(\phi \to \psi) = V(\phi) \Rightarrow_{\leq_{\mathcal{K}}} V(\psi) = \mathrm{sat}(V(\phi) \cap V(\psi)^c)^c$, and
3. $V(\Box\phi) = \Box_Q(\phi) = \{x \in X : Q(x) \subseteq V(\phi)\}$.

By a *general model* we shall mean a structure $\langle X, \mathcal{K}, Q, V\rangle$ where $\mathcal{F} = \langle X, \mathcal{K}, Q\rangle$ is an $H\Box$-frame or a general $H\Box$-frame and V is a valuation on $\mathcal{F}$. We note that a function V is a valuation in an $H\Box$-frame or a general $H\Box$-frame $\mathcal{F}$ iff it is a homomorphism between the algebra of all formulas Fm and $A(\mathcal{F})$ $(D(X))$. Then we get that a formula ϕ is valid in an $H\Box$-frame (general $H\Box$-frame) $\mathcal{F}$ iff the equation $\phi \approx 1$ is valid in the Hilbert algebra $A(\mathcal{F})$ $(D(X))$. Thus, we have that if $\mathcal{F}$ is an $H\Box$-frame (general $H\Box$-frame),

$$\mathcal{F} \vDash \phi \text{ iff } A(\mathcal{F}) \vDash \phi \approx 1 \ (D(X) \vDash \phi \approx 1).$$

Let $\mathcal{I}_{\Box}$ be an implicational modal logic. Denote by $\mathrm{Fr}(\mathcal{I}_{\Box})$ the class of all general $H\Box$-frames where the formulas of $\mathcal{I}_{\Box}$ are valid. Let $\mathrm{HSp}(\mathcal{I}_{\Box})$ be the class of all $H\Box$-spaces $\mathcal{F} = \langle X, \mathcal{K}, Q\rangle$ such that $\mathcal{F} \vDash \phi$, for all $\phi \in \mathcal{I}_{\Box}$. Clearly the class $\mathrm{HSp}(\mathcal{I}_{\Box})$ is a subclass of $\mathrm{Fr}(\mathcal{I}_{\Box})$.

We shall say that implicational modal logic $\mathcal{I}_{\Box}$ is *characterized* by a class F of general $H\Box$-frames, when $\phi \in \mathcal{I}_{\Box}$ iff ϕ is valid in every general $H\Box$-frame $\langle X, \mathcal{K}, Q\rangle \in \mathsf{F}$. Moreover, it is *frame complete* when $\phi \in \mathcal{I}_{\Box}$ iff ϕ is valid in every general $H\Box$-frame $\mathcal{F} = \langle X, \mathcal{K}, Q\rangle$, for any $\mathcal{F} \in \mathrm{Fr}(\mathcal{I}_{\Box})$. It is clear that an implicational modal logic $\mathcal{I}_{\Box}$ is frame complete if and only if it is characterized by some class of general $H\Box$-frames.

Let $\mathcal{I}_{\Box}$ be an implicational modal logic. Consider the variety of Hilbert modal algebras $\mathcal{V}(\mathcal{I}_{\Box}) = \{A \in \mathrm{Hil}_{\Box} : A \vDash \phi \approx 1, \text{ for all } \phi \in \mathcal{I}_{\Box}\}$. Simple arguments (as in classical modal logic) show that

$$\mathcal{F} \in \mathrm{HSp}(\mathcal{I}_{\Box}) \text{ iff } D(X) \in \mathcal{V}(\mathcal{I}_{\Box}).$$

Thus, we have the following result.

Proposition 5.1. *Every implicational modal logic* $\mathcal{I}_\Box$ *is characterized by the class* $\mathrm{HSp}(\mathcal{I}_\Box)$.

Let $\mathcal{F} = \langle X, \mathcal{K}, Q\rangle$ be a general $H\Box$-frame. As $D(X)$ is a subalgebra of $A(\mathcal{F})$, every formula valid in $A(\mathcal{F})$ is valid in $D(X)$, but the converse in general is not valid.

Definition 5.2. We say that the variety $\mathcal{V}$ of $H\Box$-algebras is *canonical*, if $A(\mathcal{F}(A)) \in \mathcal{V}$, when $A \in \mathcal{V}$. An implicational modal logic $\mathcal{I}_\Box$ is canonical if the variety $\mathcal{V}(\mathcal{I}_\Box)$ is canonical.

An implicational modal logic $\mathcal{I}_\Box$ is H*-persistent* if $A(\mathcal{F}) \in \mathcal{V}(\mathcal{I}_\Box)$, when $D(X) \in \mathcal{V}(\mathcal{I}_\Box)$, for every $H\Box$-space $\mathcal{F} = \langle X, \mathcal{K}, Q\rangle$.

The notion of implicational H-persistent modal logic is a generalization of the notion of d-persistent modal logic of classical modal logic (see [3] and [25]). By the results on duality between $H\Box$-spaces and modal Hilbert algebras, we can give the following result.

Proposition 5.3. *An implicational modal logic* $\mathcal{I}_\Box$ *is* H*-persistent if and only if it is canonical.*

Proof. Suppose that $\mathcal{I}_\Box$ is H-persistent. Let $A \in \mathcal{V}(\mathcal{I}_\Box)$. As A is isomorphic to $D(X(A))$, we have $D(X(A)) \in \mathcal{V}(\mathcal{I}_\Box)$. As $\mathcal{I}_\Box$ is H-persistent and taking into account that $A(\mathcal{F}((D(X(A)))$ is isomorphic to $A(\mathcal{F}(A))$, we get that $A(\mathcal{F}(A)) \in \mathcal{V}(\mathcal{I}_\Box)$. So, $\mathcal{I}_\Box$ is canonical.

For the converse we take an $H\Box$-space $\mathcal{F} = \langle X, \mathcal{K}, Q\rangle$, and suppose that $D(X) \in \mathcal{V}(\mathcal{I}_\Box)$. As $\mathcal{F}$ is an $H\Box$-space, X is homeomorphic (and also order-isomorphic) to $X(D(X))$. Then $\mathrm{Up}(X)$ is isomorphic to $\mathrm{Up}(X(D(X)))$. Thus the Hilbert modal algebras $A(\mathcal{F})$ and $A(\mathcal{F}(D(X)))$ are isomorphic, and consequently $A(\mathcal{F}) \in \mathcal{V}(\mathcal{I}_\Box)$. □

Proposition 5.4. *Every canonical implicational modal logic* $\mathcal{I}_\Box$ *is complete with respect to* $\mathrm{Fr}(\mathcal{I}_\Box)$.

Proof. The proof is as in classical modal logic. We need to prove that for each formula $\phi \notin \mathcal{I}_\Box$ there exists an $H\Box$-frame $\mathcal{F}$ of $\mathcal{I}_\Box$ such that ϕ is refuted in $\mathcal{F}$. Let $\phi \notin \mathcal{I}_\Box$. Then there exists a modal Hilbert algebra A such that $A \nvDash \phi \approx 1$. Then there exists a homomorphism $h : Fm \to A$

such that $h(\phi) \neq 1$. By Theorem 2.2 there exists $x \in X(A)$ such that $h(\phi) \notin x$. Let $\mathcal{F}(A) = \langle X(A), \mathcal{K}_A, Q_A \rangle$ be the $H\Box$-frame of A. As $\mathcal{I}_\Box$ is canonical, $A(\mathcal{F}(A)) \in \mathcal{V}(\mathcal{I}_\Box)$, i.e., $\mathcal{F}(A)$ is an $H\Box$-frame of $\mathcal{I}_\Box$. As the map $\varphi : A \to D(X(A))$ is an one to one homomorphism, the composition $\varphi \circ h$ is a homomorphism from Fm into $D(X(A))$, i.e., $\varphi \circ h$ is a valuation based on $\mathcal{F}(A)$. So, $(\varphi \circ h)(\phi) = \varphi(h(\phi)) \neq \varphi(1) = X(A)$, because $x \notin \varphi(h(\phi))$. So the formula ϕ is refuted in the general model $\langle X(A), \mathcal{K}_A, \varphi \circ h \rangle$. Therefore, ϕ is refuted in the $H\Box$-frame $\mathcal{F}(A)$. $\Box$

Given the characterizations proved in the Section 4, we can ensure that any variety of $H\Box$-algebras axiomatized by some subset of the set of equations:

$$P = \{\mathbf{S}, \mathbf{S}_n, \mathbf{T}, \mathbf{wD}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \Box 0 \to 0 \approx 1, \neg\Box a \to \Box\neg\Box a \approx 1, \Box(\Box a \to a) \approx 1\}$$

is canonical. Therefore we obtain the following result.

Theorem 5.5. *Any variety of $H\Box$-algebras axiomatized by formulas belong to P are canonical. Therefore, the associated logics are canonical and frame complete.*

6. Simple and subdirectly irreducibles $H\Box$-algebras

Denote by $\mathrm{Con}(A, \to)$ the lattice of all congruences on a Hilbert algebra A and call the set $[1]_\theta = \{x \in A : (x, 1) \in \theta\}$ the kernel of θ. If $D \in \mathrm{Fi}(A)$ then the binary relation θ_D defined by

$$(a, b) \in \theta_D \text{ iff } a \to b \in D \text{ and } b \to a \in D$$

is a congruence on A such that $[1]_{\theta_D} = D$. Moreover, the lattices $\mathrm{Fi}(A)$ and $\mathrm{Con}(A, \to)$ are isomorphic under the mutually inverse mappings $\theta \to [1]_\theta$ and $D \to \theta_D$ (see [11], [15], or [18]).

Let $A \in \mathrm{Hil}_\Box$. Denote by $\mathrm{Con}(A, \to, \Box)$ the lattice of congruences of A. Let $F \in \mathrm{Fi}(A)$. We said that F is a $\Box$-*implicative* filter if $\Box a \in F$, whenever $a \in F$, i.e., $F \subseteq \Box^{-1}(F)$. The set of all $\Box$-implicative filters of an $H\Box$-algebra A is denoted by $\mathrm{Fi}_\Box(A)$.

Let $n \in \mathbb{N}_0$. We define the symbol

$$(\alpha_n(a); b) = (a, \Box a, ..., \Box^n a; b)$$

for all $a, b \in A$. For each non-empty subset X of A, we define the set $\langle X \rangle_\Box$ as:

$$\begin{aligned} \langle X \rangle_\Box &= \{a \in A : \exists x_1, ..., x_k \in X, n_1, ..., n_k \in \mathbb{N}_0 \\ &\quad [(\alpha_{n_1}(x_1); ...; (\alpha_{n_k}(x_k); a))...) = 1]\}. \end{aligned}$$

Note that if $X = \{a\}$, then

$$\langle \{a\} \rangle_\Box = \langle a \rangle_\Box = \{b \in A : \exists n \in \mathbb{N}_0 : (\alpha_n(a); b) = 1\}.$$

Remark 6.1. As any Hilbert algebra A satisfies the Change Law, i.e., $a \to (b \to c) = b \to (a \to c)$ for all $a, b, c \in A$, we get that any $H\Box$-algebra $\langle A, \Box \rangle$ satisfies the identity

$$(\alpha_{n_1}(a); (\alpha_{n_2}(b); c)) = (\alpha_{n_2}(b); (\alpha_{n_1}(a); c))$$

for all $a, b, c \in A$, $n_1, n_2 \in \mathbb{N}_0$.

Moreover, note that if $A \in \mathrm{Hil}_\Box$ and $a, b \in A$ such that $a \le b$, then $(\alpha_n(x); a) \le (\alpha_n(x); b)$, for all $x \in A$, $n \in \mathbb{N}_0$.

Lemma 6.2. *Let $A \in \mathrm{Hil}_\Box$. Then,*

$$x \to \Box(\alpha_n(x); a) \le (\alpha_{n+1}(x); \Box a),$$

for all $x, a \in A$, $n \in \mathbb{N}_0$.

Proof. By Definition 3.1,

$$\begin{aligned} \Box(\alpha_n(x); a) &= \Box(x, \Box x, ..., \Box^n x; a) \\ &\le \Box x \to \Box(\Box x, ..., \Box^n x; a) \\ &\le \Box x \to (\Box^2 x \to (\Box^3 x \to ...(\Box^{n+1} x \to \Box a)...)). \end{aligned}$$

Thus,

$$\begin{aligned} x \to \Box(\alpha_n(x); a) &\le x \to \big(\Box x \to (\Box^2 x \to ...(\Box^{n+1} x \to \Box a)...))\big) \\ &= (\alpha_{n+1}(x); \Box a). \end{aligned}$$

$\Box$

Corollary 6.3. *Let $A \in \mathrm{Hil}_\Box$. Then,*

$$\begin{aligned} &x_k \to (x_{k-1} \to ... (x_1 \to \Box [(\alpha_{n_1}(x_1); (... (\alpha_{n_k}(x_k); a)) ...)]) ...) \le \\ &\quad \le (\alpha_{n_1+1}(x_1); (... (\alpha_{n_k+1}(x_k); \Box a)) ...) \end{aligned}$$

for all $k \in \mathbb{N}, a, x_1, ..., x_k \in A, n_1, ..., n_k \in \mathbb{N}_0$.

Proof. By Lemma 6.2,

$$x_k \to \Box(\alpha_{n_k}(x_k); a) \leq (\alpha_{n_k+1}(x_k); \Box a)\,.$$

So, by above Remark,

$$\big(\alpha_{n_{k-1}+1}(x_{k-1}); (x_k \to \Box(\alpha_k(x_k); a)\ )\big) \leq \big(\alpha_{n_{k-1}+1}(x_{k-1}); (\alpha_{n_k+1}(x_k); \Box a)\big)$$

and by Chance Law,

$$x_k \to \big(\alpha_{n_{k-1}+1}(x_{k-1}); \Box(\alpha_k(x_k); a)\big) \leq \big(\alpha_{n_{k-1}+1}(x_{k-1}); (\alpha_{n_k+1}(x_k); \Box a)\big)\,.$$

By Lemma 6.2,

$$x_{k-1} \to \Box\left(\alpha_{n_{k-1}}(x_{k-1}); (\alpha_k(x_k); a)\right) \leq \left(\alpha_{n_{k-1}+1}(x_{k-1}); \Box(\alpha_k(x_k); a)\right)\,.$$

So,

$$\begin{aligned} & x_k \to \left(x_{k-1} \to \Box\left(\alpha_{n_{k-1}}(x_{k-1}); (\alpha_k(x_k); a)\right)\right) \\ \leq \quad & x_k \to \left(\alpha_{n_{k-1}+1}(x_{k-1}); \Box(\alpha_k(x_k); a)\right) \\ \leq \quad & \left(\alpha_{n_{k-1}+1}(x_{k-1}); (\alpha_{n_k+1}(x_k); \Box a)\right)\,. \end{aligned}$$

Repeating this procedure we obtain that

$$\begin{aligned} & x_k \to (x_{k-1} \to \ldots (x_1 \to \Box\left[(\alpha_{n_1}(x_1); (\ldots (\alpha_{n_k}(x_k); a)) \ldots)\right]) \ldots) \leq \\ & \leq (\alpha_{n_1+1}(x_1); (\ldots (\alpha_{n_k+1}(x_k); \Box a)) \ldots)\,. \end{aligned}$$

$\Box$

Lemma 6.4. *Let $A \in \mathrm{Hil}_\Box$ and $X \subseteq A$. Then, $\langle X\rangle_\Box$ is the smallest $\Box$-implicative filter containing to X.*

Proof. It is clear that $\langle X\rangle_\Box \in \mathrm{Fi}(A)$. Let $a \in \langle X\rangle_\Box$. So, there exists $k \in \mathbb{N}$ and there exist $x_1, \ldots, x_k \in X, n_1, \ldots, n_k \in \mathbb{N}_0$ such that

$$(\alpha_{n_1}(x_1); (\alpha_{n_2}(x_2); \ldots((\alpha_{n_k}(x_k); a))\ldots) = 1.$$

Hence, $\Box(\alpha_{n_1}(x_1); (\alpha_{n_2}(x_2); \ldots((\alpha_{n_k}(x_k); a))\ldots) = \Box 1 = 1$. So,

$$x_k \to (x_{k-1} \to \ldots (x_1 \to \Box\,(\alpha_{n_1}(x_1); (\ldots (\alpha_{n_k}(x_k); a)) \ldots)) \ldots) = 1.$$

Thus, by above Corollary, $1 \leq (\alpha_{n_1+1}(x_1); (\ldots (\alpha_{n_k+1}(x_k); \Box a)) \ldots)$ and consequently,

$$(\alpha_{n_1+1}(x_1); (\ldots (\alpha_{n_k+1}(x_k); \Box a)) \ldots) = 1,$$

with $x_1, \ldots, x_k \in X$ and $n_1 + 1, \ldots, n_k + 1 \in \mathbb{N}_0$. Consequently, $\Box a \in \langle X \rangle_\Box$ and so, $\langle X \rangle_\Box \in \mathrm{Fi}_\Box(A)$.

Finally, it is easy to see that if $F \in \mathrm{Fi}_\Box(A)$ and $X \subseteq F$, then $\langle X \rangle_\Box \subseteq F$. □

In some subvarieties of $\mathrm{Hil}_\Box$ we can give simplified expressions of $\langle X \rangle_\Box$. If $A \in \mathrm{Hil}_\Box + \{\mathbf{4}\}$, then

$$(\alpha_n(a); b) = (\alpha_1(a); b) \tag{4}$$

for all $a, b \in A$, and for all $n \in \mathbb{N}$. If $A \in \mathrm{Hil}_\Box \mathbf{S4}$, then,

$$(\alpha_n(a); b) = \Box a \to b, \tag{5}$$

for all $a, b \in A$, and for all $n \in \mathbb{N}$.

Definition 6.5. Let $\langle X, \mathcal{K}, Q \rangle$ be an $H\Box$-space. A subset closed Y of X will be called *Q-closed* if $Q(Y) = \bigcup \{Q(y) : y \in Y\} \subseteq Y$.

The set of all Q-closed subsets of an $H\Box$-space $\langle X, \mathcal{K}, Q \rangle$ is denoted by $\mathcal{C}_Q(X)$.

If L is a lattice, L^d is the lattice with the dual order. Let L_1 and L_2 be two lattices. If two lattices L_1 and L_2 are isomorphic we write $L_1 \cong L_2$.

Proposition 6.6. *Let $A \in \mathrm{Hil}_\Box$ and let $\langle X, \mathcal{K}, Q \rangle$ be its dual space. Then,*

$$\mathrm{Con}\,(A, \to, \Box) \cong \mathrm{Fi}_\Box(A) \cong \mathcal{C}_Q(X)^d.$$

Proof. Let $\theta \in \mathrm{Con}\,(A, \to, \Box)$. It is clear that $[1]_\theta \in \mathrm{Fi}_\Box(A)$. Now, let $F \in \mathrm{Fi}_\Box(A)$. We know that $\theta_F \in \mathrm{Con}\,(A, \to)$. If $(a, b) \in \theta_F$ then $a \to b, b \to a \in F$. So, $\Box(a \to b), \Box(b \to a) \in F$. As $\Box(a \to b) \leq \Box a \to \Box b$, we get that $\Box a \to \Box b \in F$. Analogously, $\Box b \to \Box a \in F$ and so, $(\Box a, \Box b) \in \theta_F$.

We will prove that $\mathrm{Fi}_\Box(A) \cong \mathcal{C}_Q(X)^d$. Let $F \in \mathrm{Fi}_\Box(A)$. So,

$$\delta(F) = \{x \in X : F \subseteq x\} = \bigcap \{\varphi(a) \mid a \in F\},$$

is a closed subset of X. Let $y \in Q(\delta(F))$. So, exists $x \in \delta(F)$ such that $y \in Q(x)$. As F is a $\Box$-implicative filter, $F \subseteq \Box^{-1}(F) \subseteq \Box^{-1}(x) \subseteq y$, and hence, $y \in \delta(F)$. Then $\delta(F)$ is a Q-closed. Note that if $F, H \in \mathrm{Fi}_{\Box}(A)$ such that $F \subseteq H$ then $\delta(H) \subseteq \delta(F)$.

Now, we will prove that $\pi : \mathcal{C}_Q(X) \to \mathrm{Fi}_{\Box}(A)$ given by

$$\pi(Y) = \{a \in A : Y \subseteq \varphi(a)\}$$

is well-defined. It is clear that $\pi(Y) \in \mathrm{Fi}(A)$. We prove that $\pi(Y)$ is a $\Box$-implicative filter. Let $a \in A$ such that $Y \subseteq \varphi(a)$. As Y is Q-closed, $Q(Y) \subseteq Y \subseteq \varphi(a)$. Suppose that $Y \nsubseteq \varphi(\Box a)$. So, there exists $x \in Y$ such that $x \notin \varphi(\Box a)$. Thus, $\Box a \notin x$ and so, there exists $y \in X$ such that $y \in Q(x)$ and $a \notin y$. As $x \in Y$, we get $y \in Q(Y)$. Thus, $y \in Y$ and $y \notin \varphi(a)$, which is a contradiction. So, $\pi(Y) \in \mathrm{Fi}_{\Box}(A)$.

Next, we will prove that δ and π are inverses of each other. Let $Y \in \mathcal{C}_Q(X)$. So,

$$\begin{aligned} \delta(\pi(Y)) &= \bigcap \{\varphi(a) \mid a \in \pi(Y)\} &&= \bigcap \{\varphi(a) \mid Y \subseteq \varphi(a)\} \\ &= \mathrm{cl}(Y) &&= Y. \end{aligned}$$

Now, let $F \in \mathrm{Fi}_{\Box}(A)$. Suppose that there exists

$$a \in \pi(\delta(F)) = \{b \in A : \delta(F) \subseteq \varphi(b)\}$$

such that $a \notin F$, this is, $(a] \cap F = \emptyset$. By Theorem 2.2, there exists $x \in X$ such that $F \subseteq x$ and $a \notin x$, which contradicts the assumed. So, $\pi(\delta(F)) \subseteq F$. On the other hand, as $\delta(F) = \bigcap \{\varphi(a) \mid a \in F\} \subseteq \varphi(b)$ for every $b \in F$, we have that $F \subseteq \pi(\delta(F))$. Thus, we deduce that δ is a lattice anti-isomorphism. $\Box$

Let $A \in \mathrm{Hil}_{\Box}$. Let us recall that A is *subdirectly irreducible* if and only if there exists the smallest non trivial $\Box$-congruence relation θ in A. And A is *simple* if and only if A has only two $\Box$-congruence relations. By Proposition 6.6 we have that A is subdirectly irreducible iff there exists the smallest non-trivial $\Box$-implicative filter in A iff in its dual $H\Box$-space $\langle X, \mathcal{K}, Q\rangle$ there exists the largest Q-closed subset distinct from X. Moreover, A is simple iff $\mathrm{Fi}_{\Box}(A) = \{\{1\}, A\}$ iff $\mathcal{C}_Q(X) = \{\emptyset, X\}$. Now, we give a new characterization of simple and subdirectly irreducible algebras in the variety $\mathrm{Hil}_{\Box}$.

Lemma 6.7. *Let $\langle X, \mathcal{K}, Q\rangle$ be an $H\Box$-space. Then, $V_x = \mathrm{cl}(Q^*(x))$ is the smallest Q-closed set containing the element x.*

Proof. As Q^* is reflexive and $Q^*(x) \subseteq \mathrm{cl}(Q^*(x))$ for each $x \in X$, we get that $x \in \mathrm{cl}(Q^*(x))$. In adittion, as $\mathrm{cl}(Q^*(x))$ is a closed subset of X, only remains to prove that $Q(\mathrm{cl}(Q^*(x))) \subseteq \mathrm{cl}(Q^*(x))$ for each $x \in X$. Let $y \in X$ such that $y \in Q(\mathrm{cl}(Q^*(x)))$. So, there exists $z \in \mathrm{cl}(Q^*(x))$ such that $(z, y) \in Q$. Suppose that $y \notin \mathrm{cl}(Q^*(x))$, then there exists $a \in A$ such that $\mathrm{cl}(Q^*(x)) \subseteq \varphi(a)$ and $y \notin \varphi(a)$. Since $Q^*(x) \subseteq \mathrm{cl}(Q^*(x)) \subseteq \varphi(a)$, we get that $Q^n(x) \subseteq \varphi(a)$ for all $n \geq 0$. This is, $a \in w$ for all $w \in Q^n(x)$. By Lemma 4.2, $\Box^n a \in x$ for all $n \geq 0$. On the other hand, as $a \notin y$, we get that $\Box a \notin z$ and since $z \in \mathrm{cl}(Q^*(x))$, result $\varphi(\Box a)^c \cap Q^*(x) \neq \emptyset$. So, there exists $v \in X$ such that $(x, v) \in Q^m$ for some $m \geq 0$ and $\Box a \notin v$. By Lemma 4.2, $\Box^m a \notin x$ for some $m \geq 0$, which is impossible. Thus, $\mathrm{cl}(Q^*(x)) \in \mathcal{C}_Q(X)$. Let $V \in \mathcal{C}_Q(X)$ such that $x \in V$. Then $Q^n(x) \subseteq V$, for all $n \geq 0$, because V is a Q-closed. So, $Q^*(x) = \bigcup \{Q^n(x) : n \geq 0\} \subseteq V$. Thus, $\mathrm{cl}(Q^*(x)) \subseteq \mathrm{cl}(V) = V$. □

We note that $\mathrm{cl}(Q^*(x)) = \bigcap \{V : V \in \mathcal{C}_Q(X) \text{ and } x \in V\}$.

Let $\langle X, \mathcal{K}, Q \rangle$ be an $H\Box$-space. Let us define the following subsets of X:

$$I_X = \{x \in X \mid V_x = X\} \text{ and } H_X = X - I_X,$$

where $V_x = \mathrm{cl}(Q^*(x))$.

Our first main result characterizes the simple algebras as the ones of which the dual space is generated from each point.

Theorem 6.8. *Let $A \in \mathrm{Hil}_\Box$ and let $\langle X, \mathcal{K}, Q \rangle$ be its dual space. Then, the following conditions are equivalent:*

1. *A is simple,*
2. *$I_X = X$, i.e., $V_x = X$, for each $x \in X$,*
3. *$\langle a \rangle_\Box = A$, for all $a \in A - \{1\}$.*

Proof. (1) ⇒ (2) By Lemma 6.7.

(2) ⇒ (3) Suppose that there exists $a \in A - \{1\}$ such that $\langle a \rangle_\Box \neq A$. So, there exists $b \in A$ such that $b \notin \langle a \rangle_\Box$. This is, $(\alpha_n(a); b) \neq 1$ for all $n \geq 0$. So, there exists $x \in X$ such that $\Box^n a \in x$ for all $n \geq 0$ and $b \notin x$. As $\mathrm{cl}(Q^*(x)) = X$, we get that $\varphi(a)^c \cap Q^*(x) \neq \emptyset$. So, there exists $z \in Q^*(x)$ such that $a \notin z$. Hence, there exists $m \geq 0$ such that $(x, z) \in Q^m$ and $a \notin z$. By Lemma 4.2, $\Box^m a \notin x$, which is impossible.

(3) $\Rightarrow$ (1) Let $F \in \mathrm{Fi}_{\Box}(A)$. Let $a \in F$ such that $a \neq 1$. Then $\langle a \rangle_{\Box} = A \subseteq F$. Thus, $F = A$, and consequently $\mathrm{Fi}_{\Box}(A) = \{\{1\}, A\}$. Thus, A is simple. $\Box$

We note that the previous Theorem affirms that A is an $H\Box$-algebra simple if and only if $H_X = \emptyset$.

Our second main result gives a similar characterization of the subdirectly irreducible algebras.

Theorem 6.9. *Let $A \in \mathrm{Hil}_{\Box}$ and let $\langle X, \mathcal{K}, Q \rangle$ be its dual space. Then, the following conditions are equivalent:*

1. *A is subdirectly irreducible.*
2. *$H_X = \{x \in X \mid V_x \neq X\} \in \mathcal{C}_Q(X) - \{X\}$,*
3. *There exists $a \in A - \{1\}$ such that for all $b \in A - \{1\}$ there exists $n \geq 0$ such that $(\alpha_n(b); a) = 1$.*

Proof. (1) $\Rightarrow$ (2) By assumption, there exists the largest $V \in \mathcal{C}_Q(X) - \{X\}$. We will prove that $V = H_X$. It is clear that $H_X \subseteq V$. Let $x \in V$. As $V \in \mathcal{C}_Q(X)$, by Lemma 6.7, $V_x \subseteq V$. Since $V \neq X$, $V_x \neq X$ and so, $x \in H_X$.

(2) $\Rightarrow$ (3) Since $H_X \neq X$, there exists $x \in X$ such that $x \notin H_X$. As H_X is closed, there exists $a \in A - \{1\}$ such that $H_X \subseteq \varphi(a)$ and $x \notin \varphi(a)$. We will prove that for all $b \in A - \{1\}$ there exists $n \geq 0$ such that $(\alpha_n(b); a) = 1$. On the contrary, suppose that there exists $b \in A - \{1\}$ such that $(\alpha_n(b); a) \neq 1$ for all $n \geq 0$. So, there exists $w \in X$ such that $\Box^n b \in w$ for all $n \geq 0$ and $a \notin w$. As $w \notin \varphi(a)$, we get that $w \notin H_X$ and consequently, $\mathrm{cl}(Q^*(w)) = X$. Thus, $Q^*(w) \cap \varphi(b)^c \neq \emptyset$ and so, there exists $z \in Q^*(w)$ and $b \notin z$. So, there exists $m \geq 0$ such that $(w, z) \in Q^m$ and $b \notin z$. By Lemma 4.2, $\Box^m b \notin w$, which is impossible.

(3) $\Rightarrow$ (1) By assumption, $a \in \langle b \rangle_{\Box}$ for all $b \in A - \{1\}$. As $\langle b \rangle_{\Box} \in \mathrm{Fi}_{\Box}(A)$, we have that $\langle a \rangle_{\Box} \subseteq \langle b \rangle_{\Box}$ for all $b \in A - \{1\}$. As $a \neq 1$, we get that $\langle a \rangle_{\Box} \neq \{1\}$. We will prove that $\langle a \rangle_{\Box}$ is the smallest non-trivial $\Box$-implicative filter. Let $F \in \mathrm{Fi}_{\Box}(A) - \{1\}$. So, there exists $b \neq 1$ such that $b \in F$. As $\langle b \rangle_{\Box}$ is the smallest $\Box$-implicative filter containing to b, we get that $\langle a \rangle_{\Box} \subseteq \langle b \rangle_{\Box} \subseteq F$. Thus, A is subdirectly irreducible. $\Box$

Now, we shall study the simple and subdirectly irreducible algebras in the varieties $\mathrm{Hil}_{\Box}$**S4**, $\mathrm{Hil}_{\Box}^{0}$**S4**, $\mathrm{Hil}_{\Box}^{0}$**S5.1,** and $\mathrm{Hil}_{\Box}^{w}$**S5.**

Remark 6.10. Let $A \in \mathrm{Hil}_{\Box}\mathbf{S4}$ and let $\langle X, \mathcal{K}, Q\rangle$ be its dual space.

(1) By items 3 and 4 of Theorem 4.4, we get that Q is transitive and reflexive. Thus, $Q^*(x) = Q(x)$, for each $x \in X$, and as $Q(x)$ is a closed subset of X, we have that $Q(x) = V_x$, for each $x \in X$.

(2) If $H_X \neq \emptyset$, then $H_X = \bigcup\{\varphi(\Box a) : a \in A - \{1\}\}$. Indeed:

$$\begin{array}{lll} x \in H_X & \textit{iff} & Q(x) = V_x \neq X \\ & \textit{iff} & \exists y \in X : y \notin Q(x) \\ & \textit{iff} & \exists y \in X \exists a \in A : Q(x) \subseteq \varphi(a) \ \&\ y \notin \varphi(a) \\ & \textit{iff} & \exists y \in X \exists a \in A : x \in \Box_Q(\varphi(a)) = \varphi(\Box a) \ \&\ a \notin y \\ & \textit{iff} & x \in \bigcup\{\varphi(\Box a) : a \in A - \{1\}\}\,. \end{array}$$

The following result is a simple consequence of Theorem 6.8, item (1) of Remark 6.10 and the formula (5).

Proposition 6.11. *Let $A \in \mathrm{Hil}_{\Box}\mathbf{S4}$ and let $\langle X, \mathcal{K}, Q\rangle$ be its dual space. Then, the following conditions are equivalent:*

1. *A is simple.*
2. *$Q(x) = X$, for each $x \in X$.*
3. *$\langle \Box a\rangle = A$ for all $a \in A - \{1\}$. This is, A is bounded.*

Proposition 6.12. *Let $A \in \mathrm{Hil}_{\Box}\mathbf{S4}$ and let $\langle X, \mathcal{K}, Q\rangle$ be its dual space. Then, the following conditions are equivalent:*

1. *A is subdirectly irreducible.*
2. *$H_X \in D(X) - \{X\}$.*
3. *There exists $a \in A - \{1\}$ such that $\Box b \leq a$, for all $b \in A - \{1\}$.*

Proof. (1) $\Rightarrow$ (2) By Theorem 6.9, $H_X \in \mathcal{C}_Q(X) - \{X\}$. So, exists $x \in X$ such that $x \notin H_X$. Thus, there exists $c \in A - \{1\}$ such that $H_X \subseteq \varphi(c)$ and $x \notin \varphi(c)$. As in the proof of Proposition 6.6, if $H_X \in \mathcal{C}_Q(X)$ and $H_X \subseteq \varphi(c)$ then, $H_X \subseteq \varphi(\Box c)$. If $H_X \neq \emptyset$, by Remark 6.10, $H_X = \bigcup\{\varphi(\Box b) : b \in A - \{1\}\}$. As $c \neq 1$, $\varphi(\Box c) \subseteq H_X$. Thus, $H_X = \varphi(\Box c) \in D(X) - \{X\}$.

(2) ⇒ (3) Let $H_X \in D(X)-\{X\}$. So, there exists $a \in A-\{1\}$ such that $H_X = \varphi(a)$. If $H_X = \emptyset$, then $Q(x) = X$ for all $x \in X$ and by Proposition 6.11, $\langle \Box b\rangle = A$ for all $b \in A - \{1\}$. Let $a \in A - \{1\}$. Then $a \in \langle \Box b\rangle$ for all $b \in A - \{1\}$. So, $\Box b \leq a$, for all $b \in A - \{1\}$. If $H_X \neq \emptyset$, by Remark 6.10, $H_X = \bigcup\{\varphi(\Box b) : b \in A - \{1\}\} = \varphi(a)$. Therefore, $\varphi(\Box b) \subseteq \varphi(a)$ and consequently, $\Box b \leq a$ for all $b \in A - \{1\}$, because φ is an isomorphism.

(3) ⇒ (1) It is an immediate consequence of the formula (5) and Theorem 6.9. □

Corollary 6.13. *Let $A \in \mathrm{Hil}^0_\Box\mathbf{S4}$ and let $\langle X, \mathcal{K}, Q\rangle$ be its dual space. Then,*

1. *A is simple iff $\Box a = 0$, for all $a \in A - \{1\}$.*
2. *A is subdirectly irreducible iff $H_X \in D(X) - \{X\}$ iff there exists $a \in A - \{1\}$ for all $b \in A - \{1\}$ such that $\Box b \leq a$.*

Proof. (1) As A is bounded, $A = \langle 0\rangle$. Thus, by Proposition 6.11, A is simple iff $\langle \Box a\rangle = \langle 0\rangle$ for $a \in A - \{1\}$ iff $\Box a = 0$ for $a \in A - \{1\}$.

(2) By Proposition 6.12. □

Proposition 6.14. *Let $A \in \mathrm{Hil}^0_\Box\mathbf{S5.1}$. Then,*

1. *A is simple iff $\Box a = 0$, for all $a \in A - \{1\}$.*
2. *A is subdirectly irreducible not simple iff there exists $a \in A-\{1\}$ such that $\Box b \leq a$ and $\neg\Box a = 0$, for all $b \in A - \{1\}$.*

Proof. (1) By Corollary 6.13, because $\mathrm{Hil}^0_\Box\mathbf{S5.1}$ is subvariety of $\mathrm{Hil}^0_\Box\mathbf{S4}$.

(2) Let A be subdirectly irreducible. So, there exists $a \in A - \{1\}$ such that $\Box b \leq a$, for all $b \in A - \{1\}$. It remains to prove that A is not simple iff $\neg\Box a = 0$. If A is not simple then exists $b \neq 1$ such that $\Box b \neq 0$, i.e., $\Box b \nleq 0$. This is, $\neg\Box b \neq 1$ and so, $\Box\neg\Box b \leq a$. Thus, $\neg\Box b \leq a$ and hence, $\neg\Box a \leq \neg\neg\Box b$. As any Hilbert algebra A satisfies $(c \to d) \to ((d \to c) \to c) = (d \to c) \to ((c \to d) \to d)$, replacing c by 0 result $\neg\neg d = \neg d \to d$. Thus, $\neg\Box a \leq \neg\Box b \to \Box b \leq \neg\Box b \to b$ and so, $\neg\Box a \to (\neg\Box b \to b) = (\neg\Box a \to \neg\Box b) \to (\neg\Box a \to b) = 1$. As $b \neq 1$, we have $\Box b \leq a$ and so, $\Box b = \Box^2 b \leq \Box a$. Thus, $\neg\Box a \to \neg\Box b = 1$ and consequently, $\neg\Box a \to b = 1$. As $\neg\Box a \leq b \neq 1$, we get that $\neg\Box a \neq 1$ and so, $\neg\Box a \leq \Box\neg\Box a \leq a$.

Hence, $(\alpha_0(\neg\Box a); a) = 1$ and thus, $a \in \langle\neg\Box a\rangle_\Box$. As $\langle\neg\Box a\rangle_\Box \in \mathrm{Fi}_\Box(A)$, $\Box a \in \langle\neg\Box a\rangle_\Box$ and so, $0 \in \langle\neg\Box a\rangle_\Box$. Thus, $\neg\Box a = 0$. Reciprocally, if there exists $a \neq 1$ such that $\neg\Box a = 0$, then $\Box a \to 0 \neq 1$. This is, $\Box a \not\leq 0$ and so, $\Box a \neq 0$. Thus, A is not simple. □

Lemma 6.15. *Let $A \in \mathrm{Hil}^w_\Box\mathbf{S5}$. Then, $\langle a\rangle_\Box = \{b : a \to (\Box a \to b) = 1\}$.*

Proof. It is easy and left to the reader. □

Proposition 6.16. *Let $A \in \mathrm{Hil}^w_\Box\mathbf{S5}$. Then,*

1. *A is simple iff $\Box a = 0$, for all $a \in A - \{1\}$.*
2. *A is subdirectly irreducible iff there exists $a \in A - \{1\}$ such that $(\alpha_1(b); a) = 1$ for all $b \in A - \{1\}$.*

Proof. Let $A \in \mathrm{Hil}^w_\Box\mathbf{S5}$. By Remark 4.1, $\Box a \leq \Box^2 a$ for all $a \in A$.

1. ($\Rightarrow$) Let $a \in A$. As $\Box a \leq \Box^2 a$, we get that $\Box b \in \langle\Box a\rangle$ when $b \in \langle\Box a\rangle$. Thus, $\langle\Box a\rangle \in \mathrm{Fi}_\Box(A)$. As A is simple, $\langle\Box a\rangle = A$ or $\langle\Box a\rangle = \{1\}$. This is, $\Box a = 0$ or $\Box a = 1$. The proof is completed by showing that $\Box a = 1$ iff $a = 1$. Suppose that there exists $a \neq 1$ such that $\Box a = 1$. As A is simple, by Theorem 6.8, $\langle a\rangle_\Box = A$. Note that $\langle a\rangle_\Box = \langle a\rangle$. In fact, it is clear that $\langle a\rangle \subseteq \langle a\rangle_\Box$. Let $b \in \langle a\rangle_\Box$. By Lemma 6.15 we have $1 = a \to (\Box a \to b) = a \to (1 \to b) = a \to b$. So, $b \in \langle a\rangle$. Thus, $A = \langle a\rangle$, and consequently $a = 0$. Thus, $\Box a = 0$ which is impossible.

($\Leftarrow$) It is clear that $\Box a \in \langle a\rangle_\Box$. So, $\langle\Box a\rangle \subseteq \langle a\rangle_\Box$ for all $a \in A$. By assumption, $A = \langle 0\rangle = \langle\Box a\rangle \subseteq \langle a\rangle_\Box$ for $a \in A - \{1\}$ and consequently $A = \langle a\rangle_\Box$, for $a \in A - \{1\}$. Then by Theorem 6.8, A is simple.

2. By Theorem 6.9, there exists $a \in A - \{1\}$ such that for all $b \in A - \{1\}$ there exists $n \geq 0$ such that $(\alpha_n(b); a) = 1$. So, $(\alpha_0(b); a) = 1$ or $(\alpha_n(b); a) = 1$ for $n \in \mathbb{N}$. By (4), $b \leq a$ or $(\alpha_1(b); a) = 1$. If $b \leq a$, as $a \leq \Box b \to a$, result that $b \leq \Box b \to a$ and so, $(\alpha_1(b); a) = 1$. The converse is an immediate consequence of Theorem 6.9. □

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