REPORTS ON MATHEMATICAL LOGIC
48 (2013), 81–100
DOI:10.4467/20842589RM.13.004.1256

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A SEMANTIC ANALYSIS OF SOME DISTRIBUTIVE LOGICS WITH NEGATION

A b s t r a c t. In this paper we shall study some extensions of the semilattice based deductive systems $\mathcal{S}(\mathsf{N})$ and $\mathcal{S}(\mathsf{N},1)$, where N is the variety of bounded distributive lattices with a negation operator. We shall prove that $\mathcal{S}(\mathsf{N})$ and $\mathcal{S}(\mathsf{N},1)$ are the deductive systems generated by the local consequence relation and the global consequence relation associated with $\neg$-frames, respectively. Using algebraic and relational methods we will prove that $\mathcal{S}(\mathsf{N})$ and some of its extensions are canonical and frame complete.

1. Introduction

In [8] and [9] Došen introduces negation as a kind of impossibility operator (or non-necessity modal operator) in intuitionistic logics. In [10] J. Michael

Received 29 March 2012
We would like to thank the referees for the comments and suggestions on the paper. This research was supported by the CONICET under grant PIP 112-200801-02543.

Dunn and C. Zhou give a detailed analysis of different negations in distributive logics. Some of these negations can be treated as a generalization of the intuitionistic negation or as a generalization of a dual intuitionistic negation (as for instance in [16]). Another interesting propositional logic with negation is the Subminimal logic introduced recently in [2]. This logic is based on the implicationless fragment of Johansson's propositional logic.

In [5] it was introduced the variety N of distributive lattices with a negation operator, or $\neg$-lattices, as a generalization of some algebraic structures like Boolean algebras, pseudocomplemented distributive lattices [3], and quasi-Stone algebras [15]. The variety of $\neg$-lattices (called bounded distributive lattices with a negative modal operator in [10]) is the algebraic interpretation of the calculus K_i of the preminimal negation defined in [10], and it is also just the $\{\vee, \wedge, \neg, \bot, \top\}$-fragment of Dosěn's system N [8]. In [13] S. P. Odintsov studied the logic N^* (an axiomatic extension of N). The adequate algebraic semantics for N^* is the variety of Heyting algebras endowed with an Ockham negation.

In [12] (see also [11]) R. Jansana develops the theory of the selfextensional logics with a conjunction $\wedge$. A variety of algebras V of algebraic type $\mathcal{L}$ is called a *semilattice class relative to* $\wedge$ if for every $\boldsymbol{A} \in \mathsf{V}$ the reduct $\langle A, \wedge, 1\rangle$ is a $\wedge$-semilattice with top element 1. For any of these varieties it is possible to define the deductive system $\mathcal{S}(\mathsf{V})$, called the *semilattice based deductive system relative to* $\wedge$ and V, and the deductive system $\mathcal{S}(\mathsf{V}, 1)$, called the *assertional logic* of V. Since each $\neg$-lattice has a $\wedge$-semilattice reduct, we can use the variety N to define the deductive systems $\mathcal{S}(\mathsf{N})$ and $\mathcal{S}(\mathsf{N}, 1)$. The aim of this paper is to study these deductive systems and some of their extensions using algebraic and relational methods.

The paper is divided in six sections. In the second section we shall introduce all the preliminary notions and results relevant to the paper. In the third section we will define the basis deductive system $\mathcal{S}_\neg$. This deductive system is essentially the same as the logical system K_i defined by J. Michael Dunn and C. Zhou in [10]. From the general results given by Jansana in [12] we have that $\mathcal{S}_\neg$ is also the deductive system $\mathcal{S}(\mathsf{N})$, i.e., $\mathcal{S}_\neg = \mathcal{S}(\mathsf{N})$.

In Section 4 we will define the frames for the deductive system $\mathcal{S}(\mathsf{N})$, called $\neg$-frames. These frames were first defined by K. Dosěn in [9], and also were used in the representation theory developed in [5] and [6] for distributive lattices with a negation operator (see also [7] for related results). We shall prove that the deductive systems $\mathcal{S}(\mathsf{N})$ and $\mathcal{S}(\mathsf{N},1)$ are the deductive system generated by the local consequence relation and the global consequence relation, respectively, associated with the models of the $\{\vee, \wedge, \neg, \bot, \top\}$-fragment of Dosěn's system N. In Section 5 we will consider some sequents which correspond to first-order conditions on $\neg$-frames. A variety $\mathcal{V}$ of $\neg$-lattices is *canonical* if for any $\boldsymbol{A} \in \mathcal{V}$, we have that the canonical extension $\boldsymbol{A}(\mathcal{F}(\boldsymbol{A}))$ belongs to $\mathcal{V}$, where $\mathcal{F}(\boldsymbol{A})$ is the $\neg$-frame of $\boldsymbol{A}$. The notion of canonical varieties is the algebraic interpretation of canonical modal logics (see [4]). It is easy to see that the variety N is canonical. In Section 6 we will prove that the varieties WA, QS, WQS, and PA are also canonical, and thus the associate semilattice deductive systems are canonical and frame complete.

2. Preliminaries

Given $\mathcal{L}$ an algebraic similarity type, we will consider $\boldsymbol{Fm}$ the absolutely free algebra of type $\mathcal{L}$ with a denumerable set of generators, called *propositional variables.* The elements of $\boldsymbol{Fm}$ are called *formulas.* All algebras considered will be of this type. The set of all homomorphisms from an algebra $\boldsymbol{A}$ to an algebra $\boldsymbol{B}$ is denoted by $\mathrm{Hom}(\boldsymbol{A}, \boldsymbol{B})$. A *substitution* is a homomorphism from $\boldsymbol{Fm}$ into itself. A *(finitary) logic or deductive system* of type $\mathcal{L}$ is a pair $\mathcal{S} = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}} \rangle$, where $\vdash_{\mathcal{S}} \subseteq \mathcal{P}(\boldsymbol{Fm}) \times \boldsymbol{Fm}$ is a standard consequence relation between sets of formulas and formulas, i.e., it is a finitary consequence relation invariant under substitutions. A *sequent* of type $\mathcal{L}$ will be a pair $\langle \Gamma, \varphi \rangle$ where Γ is a finite subset of $\boldsymbol{Fm}$ and φ is a formula in $\boldsymbol{Fm}$. Usually, we will note a sequent $\langle \Gamma, \varphi \rangle$ by $\Gamma \vdash \varphi$, and we will say that a sequent $\Gamma \vdash \varphi$ is a sequent of $\mathcal{S}$ if $\Gamma \vdash_{\mathcal{S}} \varphi$. We say that a formula φ is *deducible* in a deductive system $\mathcal{S}$ from a set of formulas Δ, in symbols $\Delta \vdash_{\mathcal{S}} \varphi$, if there is a *finite* set of formulas $\Gamma \subseteq \Delta$ such that the sequent $\Gamma \vdash \varphi$ belongs to $\mathcal{S}$. A deductive system $\mathcal{S}'$ is an *extension* of a deductive system $\mathcal{S}$ if $\mathcal{S} \subseteq \mathcal{S}'$.

A deductive system $\mathcal{S}$ is *selfextensional*, that is, its relation of interderivability $\dashv\vdash_{\mathcal{S}}$ defined by $\varphi \dashv\vdash_{\mathcal{S}} \phi$ iff $\varphi \vdash_{\mathcal{S}} \phi$ and $\phi \vdash_{\mathcal{S}} \varphi$ is a congruence in the formula algebra $\boldsymbol{Fm}$.

Let $\mathcal{S}$ be a deductive system over an algebraic type $\mathcal{L}$. A binary connective $\wedge \in \mathcal{L}$ is called a *conjunction* of $\mathcal{S}$ if for every formula φ, ψ the following conditions hold:

$$\varphi, \psi \vdash_{\mathcal{S}} \varphi \wedge \psi \qquad \varphi \wedge \psi \vdash_{\mathcal{S}} \varphi, \qquad \varphi \wedge \psi \vdash_{\mathcal{S}} \psi.$$

A class of algebras V of an algebraic type $\mathcal{L}$ is called a *semilattice class relative to* $\wedge$ if for every algebra $\boldsymbol{A} \in \mathsf{V}$ the reduct $\langle A, \wedge \rangle$ is a $\wedge$-semilattice. We suppose that *every* algebra in the semilattice based class V has a top element 1. We use the symbol $\approx$ to represent formal equations. If the equation $\phi \approx \varphi$ is valid an algebra $\boldsymbol{A}$ we write $\boldsymbol{A} \vDash \phi \approx \varphi$. We shall say that a sequent $\{\varphi_1, ..., \varphi_n\} \vdash \varphi$ is *valid in* an algebra $\boldsymbol{A}$ iff $\forall h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$, $h(\varphi_1) \wedge \ldots \wedge h(\varphi_n) \leq h(\varphi)$, i.e., $\boldsymbol{A} \vDash \varphi \wedge ... \wedge \varphi_n \wedge \varphi \approx \varphi \wedge ... \wedge \varphi_n$. We write $\mathsf{V} \vDash \phi \approx \varphi$, when $\boldsymbol{A} \vDash \phi \approx \varphi$ for all $\boldsymbol{A} \in \mathsf{V}$.

Let V be a semilattice class relative to $\wedge$ such that it is a variety. Now we shall define a deductive system $\mathcal{S}(\mathsf{V})$ as:

$$\begin{aligned} &\varphi_1, ..., \varphi_n \vdash_{\mathcal{S}(\mathsf{V})} \varphi \quad \text{iff} \quad \mathsf{V} \vDash \varphi \wedge ... \wedge \varphi_n \wedge \varphi \approx \varphi \wedge ... \wedge \varphi_n, \\ &\emptyset \vdash_{\mathcal{S}(\mathsf{V})} \varphi \qquad\qquad \text{iff} \quad \mathsf{V} \vDash \varphi \approx 1. \end{aligned} \tag{2.1}$$

In [12], the author proves that if V is a variety, then the reduced generalized matrices of $\mathcal{S}(\mathsf{V})$ are exactly the members of V, and that the filters of the logic $\mathcal{S}(\mathsf{V})$ in each algebra in V are the semilattice filters of the algebra (plus the empty set if the logic $\mathcal{S}(\mathsf{V})$ do not have theorems). The deductive system $\mathcal{S}(\mathsf{V})$ is called the *semilattice based deductive system relative to* $\wedge$ *and* V.

The other deductive system is the *assertional logic* $\mathcal{S}(\mathsf{V}, 1)$, also *called the logic that preserves truth with respect to the class* V (where truth is represented by the constant 1) (see [1] or [11]). This logic $\mathcal{S}(\mathsf{V}, 1)$ is defined by:

$$\begin{aligned} &\varphi_1, ..., \varphi_n \vdash_{\mathcal{S}(\mathsf{V},1)} \varphi \quad \text{iff} \quad \mathsf{V} \vDash \varphi_1 \approx 1 \ \& \ \cdots \ \& \ \varphi_n \approx 1 \Rightarrow \varphi \approx 1 \\ &\emptyset \vdash_{\mathcal{S}(\mathsf{V},1)} \varphi \qquad\qquad \text{iff} \quad \mathsf{V} \vDash \varphi \approx 1. \end{aligned}$$

Since $\{1_{\boldsymbol{A}}\}$ is a semilattice filter for every $\boldsymbol{A}$ in V, $\mathcal{S}(\mathsf{V}, 1)$ is an extension of $\mathcal{S}(\mathsf{V})$. By the results given in [12] we have $\mathcal{S}(\mathsf{V}) = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}(\mathsf{V})} \rangle$ and $\mathcal{S}(\mathsf{V}, 1) = \langle \boldsymbol{Fm}, \vdash_{\mathcal{S}(\mathsf{V},1)} \rangle$ are (finitary) deductive systems.

We note that the deductive system $\mathcal{S}(\mathsf{V})$ is *selfextensional.* Moreover, $\varphi \dashv\vdash_{\mathcal{S}(\mathsf{V})} \phi$ iff $\mathsf{V} \vDash \varphi \approx \phi$ (see Lemma 2 of [11]).

A *negated lattice*, or *¬-lattice or bounded distributive lattice with a negative modal operator* [5], is an algebra $\boldsymbol{A} = \langle A, \neg \rangle$, where A is a bounded distributive lattice, and $\neg$ is a unary operation defined on A such that

N1 $\neg(a \vee b) = \neg a \wedge \neg b$,

N2 $\neg 0 = 1$.

A ¬-lattice $\boldsymbol{A}$ is a *bounded weak-algebra* if it satisfies the equations:

W1 $a \wedge \neg(a \wedge b) \leq \neg b$,

W2 $a \leq \neg\neg a$.

A *quasi-Stone algebra* (QS-algebra) [15] is a ¬-lattice $\boldsymbol{A}$ satisfying the following equations

Q1 $a \wedge \neg\neg a = a$,

Q2 $\neg a \vee \neg\neg a = 1$.

A *weak-quasi-Stone algebra* (WQS-algebra) [7] is a ¬-lattice $\boldsymbol{A}$ satisfying the following conditions:

WQ1 $\neg a \wedge \neg\neg a = 0$,

WQ2 $\neg a \vee \neg\neg a = 1$.

A *pseudocomplemented distributive lattice* (or *p-algebra*) [3] is a pair $\langle A, \neg \rangle$ where A is a bounded distributive lattice and $\neg$ is a unary operation on A satisfying the following identities:

SP1 $a \wedge \neg(a \wedge b) = a \wedge \neg b$,

SP2 $a \wedge \neg 0 = a$,

SP3 $\neg\neg 0 = 0$.

The variety of quasi-Stone algebras was introduced in [15]. The variety of weak-quasi-Stone algebra was defined and studied in [7] as a natural generalization of the quasi-Stone algebras. The variety of weak-algebras without zero was introduced in [2]. The varieties of ¬-lattices, bounded weak-algebras, quasi-Stone algebras, weak-quasi-Stone algebras, and p-algebras are denoted by N, WA, QS, WQS, and PA, respectively. All these varieties are semilattice classes relative to $\wedge$. Thus we can consider the deductive systems $\mathcal{S}(\mathsf{N})$, $\mathcal{S}(\mathsf{WA})$, $\mathcal{S}(\mathsf{QS})$, $\mathcal{S}(\mathsf{WQS})$, and $\mathcal{S}(\mathsf{PA})$.

3. The basic deductive system $\mathcal{S}_\neg$

A *distributive lattice logic with* a *negation* is a binary consequence system $\vdash\subseteq \boldsymbol{Fm}\times\boldsymbol{Fm}$ in the language $\{\vee,\wedge,\neg,\bot,\top\}$ containing the following postulates and rules:

$$\varphi\vdash\varphi \qquad \varphi\vdash\top \qquad \bot\vdash\varphi \qquad \top\vdash\neg\bot$$

$$\varphi\wedge\psi\vdash\varphi \qquad \varphi\wedge\psi\vdash\psi \qquad \varphi\vdash\varphi\vee\psi \qquad \psi\vdash\varphi\vee\psi$$

$$\varphi\wedge(\psi\vee\alpha)\vdash(\varphi\wedge\psi)\vee(\varphi\wedge\alpha) \qquad \frac{\varphi\vdash\psi \quad \varphi\vdash\alpha}{\varphi\vdash\psi\wedge\alpha} \qquad \frac{\varphi\vdash\alpha \quad \psi\vdash\alpha}{\varphi\vee\psi\vdash\alpha}$$

$$\frac{\varphi\vdash\psi}{\neg\psi\vdash\neg\varphi} \qquad \neg\varphi\wedge\neg\psi\vdash\neg(\varphi\vee\psi)$$

We shall say that a sequent $\{\varphi_1,...,\varphi_n\}\vdash\varphi$ is *derivable* iff the pair $\varphi_1\wedge...\wedge\varphi_n\vdash\varphi$ is derivable by means of the previous sequents and rules. Define a deductive system $\mathcal{S}_\neg=\langle\boldsymbol{Fm},\vdash_{\mathcal{S}_\neg}\rangle$ as follows:

$$\Gamma\vdash_{\mathcal{S}_\neg}\varphi \text{ iff } \exists\{\varphi_1,...,\varphi_n\}\subseteq\Gamma \text{ such that } \{\varphi_1,...,\varphi_n\}\vdash\varphi \text{ is derivable.}$$

We note that $\{\varphi_1,...,\varphi_n\}\vdash_{\mathcal{S}_\neg}\varphi$ iff $\varphi_1\wedge...\wedge\varphi_n\vdash_{K_i}\varphi$ is a consequence pair in the logical system K_i defined in [10]. It is easy to see that the system K_i is essentially the deductive system $\mathcal{S}_\neg$. We note that $S_\neg$ is also the $\{\vee,\wedge,\neg,\bot,\top\}$-fragment of Dosěn's system N from [8].

It is clear that the deductive system $\mathcal{S}_\neg$ is selfextensional, because relation of interderivability $\dashv\vdash_{\mathcal{S}_\neg}$ given by $\varphi\dashv\vdash_{\mathcal{S}_\neg}\phi$ iff $\varphi\vdash_{\mathcal{S}_\neg}\phi$ and $\phi\vdash_{\mathcal{S}_\neg}\varphi$ is a congruence in the formula algebra $\boldsymbol{Fm}$. Let $\boldsymbol{A}_{\mathcal{S}_\neg}=\boldsymbol{Fm}/\dashv\vdash_{\mathcal{S}_\neg}$ be the *Lindenbaum-Tarski* algebra of $\mathcal{S}_\neg$. It is immediate to see that $\boldsymbol{A}_{\mathcal{S}_\neg}$ is a $\neg$-lattice. Let $\pi:\boldsymbol{Fm}\to\boldsymbol{A}_{\mathcal{S}_\neg}$ be the canonical projection homomorphism. It is clear that $\varphi\vdash_{\mathcal{S}_\neg}\phi$ iff $\pi(\varphi)\wedge\pi(\phi)=\pi(\varphi)$ iff $\boldsymbol{A}_{\mathcal{S}_\neg}\vDash\varphi\wedge\phi\approx\varphi$. Thus, $\varphi\dashv\vdash_{\mathcal{S}_\neg}\phi$ iff $\pi(\varphi)=\pi(\phi)$ iff $\boldsymbol{A}_{\mathcal{S}_\neg}\vDash\varphi\approx\phi$.

Let $\mathcal{S}(\mathsf{N})$ be the semilattice based deductive system relative to $\wedge$ and N. Now we can give the following algebraic completeness of $\mathcal{S}_\neg$.

Theorem 3.1 (Algebraic completeness)**.** *Let* N *be the variety of all* $\neg$-*lattices. Then* $\Gamma\vdash_{\mathcal{S}_\neg}\varphi$ *iff* $\Gamma\vdash_{\mathcal{S}(\mathsf{N})}\varphi$*, for any set of formulas* Γ *and any formula* φ*, i.e.,* $\mathcal{S}_\neg=\mathcal{S}(\mathsf{N})$*.*

Proof. Let $\Gamma \cup \{\varphi\}$ be a set of formulas. Assume that $\Gamma \vdash_{\mathcal{S}_\neg} \varphi$. Then there exists a finite subset $\{\varphi_1, ..., \varphi_n\}$ of Γ such that $\varphi_1, ..., \varphi_n \vdash_{\mathcal{S}_\neg} \varphi$, i.e., $\varphi_1 \wedge ... \wedge \varphi_n \vdash_{\mathcal{S}_\neg} \varphi$. So, $\boldsymbol{A}_{\mathcal{S}_\neg} \vDash \varphi_1 \wedge ... \wedge \varphi_n \wedge \varphi \approx \varphi_1 \wedge ... \wedge \varphi_n$. We prove that $\varphi_1, ..., \varphi_n \vdash_{\mathcal{S}(\mathsf{N})} \varphi$. Let $\boldsymbol{A} \in \mathsf{N}$. Let $h \in \mathrm{Hom}(\boldsymbol{Fm}, \boldsymbol{A})$. We consider the homomorphism $\bar{h} : \boldsymbol{A}_{\mathcal{S}_\neg} \to \boldsymbol{A}$ defined by $\bar{h}(\pi(\varphi)) = h(\varphi)$, for $\varphi \in \boldsymbol{Fm}$, where $\pi : \boldsymbol{Fm} \to \boldsymbol{A}_{\mathcal{S}_\neg}$ is the canonical projection map. So, $\boldsymbol{A}_{\mathcal{S}_\neg} \vDash \varphi_1 \wedge ... \wedge \varphi_n \wedge \varphi \approx \varphi_1 \wedge ... \wedge \varphi_n$ iff $\pi(\varphi_1 \wedge ... \wedge \varphi_n \wedge \varphi) = \pi(\varphi_1 \wedge ... \wedge \varphi_n)$. So, $\bar{h}(\pi(\varphi_1 \wedge ... \wedge \varphi_n \wedge \varphi)) = \bar{h}(\pi(\varphi_1 \wedge ... \wedge \varphi_n))$, i.e., $h(\varphi_1 \wedge ... \wedge \varphi_n \wedge \varphi) = h(\varphi_1 \wedge ... \wedge \varphi_n)$. Thus, $\boldsymbol{A} \vDash \varphi_1 \wedge ... \wedge \varphi_n \wedge \varphi \approx \varphi_1 \wedge ... \wedge \varphi_n$, and consequently $\mathsf{N} \vDash \varphi \wedge ... \wedge \varphi_n \wedge \varphi \approx \varphi \wedge ... \wedge \varphi_n$. Therefore, $\varphi_1, ..., \varphi_n \vdash_{\mathcal{S}(\mathsf{N})} \varphi$, i.e., $\Gamma \vdash_{\mathcal{S}(\mathsf{N})} \varphi$.

Assume that $\Gamma \vdash_{\mathcal{S}(\mathsf{N})} \varphi$. Then there exists a finite subset $\{\varphi_1, ..., \varphi_n\}$ of Γ such that $\varphi_1, ..., \varphi_n \vdash_{\mathcal{S}(\mathsf{N})} \varphi$, i.e., $\mathsf{N} \vDash \varphi \wedge ... \wedge \varphi_n \wedge \varphi \approx \varphi \wedge ... \wedge \varphi_n$. As $\boldsymbol{A}_{\mathcal{S}_\neg}$ is a $\neg$-lattice, we have $\boldsymbol{A}_{\mathcal{S}_\neg} \vDash \varphi \wedge ... \wedge \varphi_n \wedge \varphi \approx \varphi \wedge ... \wedge \varphi_n$. Then, $\varphi_1 \wedge ... \wedge \varphi_n \vdash_{\mathcal{S}_\neg} \varphi$, and thus we get that $\varphi_1, ..., \varphi_n \vdash_{\mathcal{S}_\neg} \varphi$, i.e., $\Gamma \vdash_{\mathcal{S}_\neg} \varphi$. $\square$

We shall now introduce several sequents that we will used to define extensions of the basic deductive system $\mathcal{S}(\mathsf{N})$. Some of these extensions have been considered by J. Michael Dunn and Chunlai Zhou in [10].

Given a deductive system $\mathcal{S}$ and a set of sequents $\{\Gamma_i \vdash \varphi_i : i \in I\}$, we denote by $\mathcal{S} \cup \{\Gamma_i \vdash \varphi_i : i \in I\}$ the least deductive system $\mathcal{S}'$ that extends $\mathcal{S}$ and for each $i \in I$ and for any substitution instance of $\Gamma_i \vdash \varphi_i$ belong to $\mathcal{S}'$. Let us consider the following sequents:

(1)	$\top \vdash \neg(\varphi \wedge \neg\varphi)$	**(5)**	$\neg\varphi \wedge \varphi \vdash \bot$
(2)	$\varphi \vdash \neg\neg\varphi$	**(6)**	$\varphi \wedge \neg(\varphi \wedge \psi) \vdash \neg\psi$
(2)	$\vdash \neg\varphi \vee \neg\neg\varphi$	**(7)**	$\neg\varphi \wedge \neg\neg\varphi \vdash \bot$
(4)	$\varphi \wedge \neg\varphi \vdash \neg\psi$	**(8)**	$\neg\top \vdash \bot$.

From Theorem 3.1, and taking into account that for each sequent $\varphi \vdash \psi$ we can consider the identity $\varphi \wedge \psi \approx \psi$, we can affirm that for extension of the deductive system $\mathcal{S}(\mathsf{N})$ by means of some subset of the set of sequents $\{\mathbf{(1)}, \mathbf{(2)}, \mathbf{(3)}, \mathbf{(4)}, \mathbf{(5)}, \mathbf{(6)}, \mathbf{(7)}, \mathbf{(8)}\}$ is complete with respect to the corresponding variety of ¬-lattices. In particular we have that:

$$\begin{aligned}
\mathcal{S}(\mathsf{WA}) &= \mathcal{S}(\mathsf{N}) \cup \{\varphi \wedge \neg(\varphi \wedge \psi) \vdash \neg\psi, \varphi \vdash \neg\neg\varphi\}, \\
\mathcal{S}(\mathsf{QS}) &= \mathcal{S}(\mathsf{N}) \cup \{\varphi \vdash \neg\neg\varphi, \top \vdash \neg\varphi \vee \neg\neg\varphi\}, \\
\mathcal{S}(\mathsf{WQS}) &= \mathcal{S}(\mathsf{N}) \cup \{\neg\varphi \wedge \neg\neg\varphi \vdash \bot, \top \vdash \neg\varphi \vee \neg\neg\varphi\}, \\
\mathcal{S}(\mathsf{PA}) &= \mathcal{S}(\mathsf{WA}) \cup \{\neg\varphi \wedge \varphi \vdash \bot\}.
\end{aligned}$$

The proofs of the following results are easy and left to the reader.

Lemma 3.2. *(1) The deductive systems* $\mathcal{S}(\mathsf{N}) \cup \{\varphi \wedge \neg(\varphi \wedge \psi) \vdash \neg\psi\}$ *is equivalent to the deductive system* $\mathcal{S}(\mathsf{N})$ *with the rule* (**ANT**) *(antilogism)* $\dfrac{\varphi, \psi \vdash \alpha}{\varphi, \neg\alpha \vdash \neg\psi}$.

(2) The deductive system $\mathcal{S}(\mathsf{PA})$ *is equivalent to the deductive system* $\mathcal{S}(\mathsf{N})$ *with the rules* (**ANT**) *and* (**NI**) $\dfrac{\varphi \vdash \alpha}{\varphi, \neg\alpha \vdash \bot}$.

Proof. We only prove (1). By Theorem 3.1 we can give an algebraic proof. Let $\boldsymbol{A}$ be a ¬-lattice. Let $a, b, c \in A$. Suppose that $a \wedge b \leq c$. Then $\neg c \leq \neg(a \wedge b)$. So, $a \wedge \neg c \leq a \wedge \neg(a \wedge b) \leq \neg b$.

We prove the converse. As $a \wedge b \leq a \wedge b$, we get $a \wedge \neg(a \wedge b) \leq \neg b$. □

4. Relational semantics and Completeness

Let $\langle X, \leq \rangle$ be a poset. A subset Y of X is increasing if for every $x \in Y$ and for all $y \in X$, if $x \leq y$ it holds that $y \in Y$. The power set of a set X will be denoted by $\mathcal{P}(X)$. The set of all *increasing subsets* of X will be denoted by $\mathcal{P}_i(X)$. Given a binary relation R on a set X, let $R(x) = \{y \in X \mid (x, y) \in R\}$, for $x \in X$. The composition between two relations R, and S of X is the relation $R \circ S = \{(x, y) \mid \exists z \in X((x, z) \in R \text{ and } (z, y) \in S\}$. Define the operator $\neg_R$ as

$$\neg_R(U) = \{x \in X \mid R(x) \cap U = \emptyset\},$$

for each $U \subseteq X$. We consider the auxiliary relation $R_\neg$ defined by $R_\neg = R \circ \leq^{-1}$. The proofs of the following results can be found in [5], or [6].

Proposition 4.1. *Let* $\langle X, \leq \rangle$ *be a poset and let* R *be a binary relation on* X*. Then*

1. *$\neg_R(U) = \neg_{R\circ\leq^{-1}}(U)$, for all $U \in \mathcal{P}_i(X)$.*

2. *$\leq \circ R \subseteq R \circ \leq^{-1}$ iff $\neg_R(U) \in \mathcal{P}_i(X)$, for all $U \in \mathcal{P}_i(X)$.*

Definition 4.2. A *compatibility frame, or $\neg$-frame,* is a relational structure $\mathcal{F} = \langle X, \leq, R\rangle$ where $\langle X, \leq\rangle$ is a poset, and R is a binary relation on X such that $\leq \circ R \subseteq R \circ \leq^{-1}$.

By Proposition 4.1 we have that the structure $\mathcal{F} = \langle X, \leq, R\rangle$ is a $\neg$-*frame iff* $\neg_R(U) \in \mathcal{P}_i(X)$, i.e., $\boldsymbol{A}(\mathcal{F}) = \langle \mathcal{P}_i(X), \neg_R\rangle$ is a $\neg$-lattice.

A *valuation* on a frame $\mathcal{F} = \langle X, \leq, R\rangle$ is a function $V : Var \to \mathcal{P}_i(X)$. A valuation V can be extended recursively to the set of all formulas $\boldsymbol{Fm}$ by means of the following clauses:

1. $V(\top) = X$,

2. $V(\bot) = \emptyset$,

3. $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$,

4. $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$,

5. $V(\neg\varphi) = \{x \in X \mid R(x) \cap V(\varphi) = \emptyset\}$.

A *model* is a pair $\mathcal{M} = \langle \mathcal{F}, V\rangle$, where $\mathcal{F}$ is a $\neg$-*frame* and V is a valuation on it. By induction and using the condition $\leq \circ R \subseteq R\circ \leq^{-1}$ we can to prove that $V(\varphi) \in \mathcal{P}_i(X)$, for all $\varphi \in \boldsymbol{Fm}$. From Proposition 4.1 we deduce that $V(\neg\varphi) = \neg_{R_\neg}(V(\varphi))$, for each formula φ. A formula φ is *valid* in a frame $\mathcal{F}$, in symbols $\mathcal{F} \vDash \varphi$, if $V(\varphi) = X$, for all valuation V defined on it. We note that a function $V : \boldsymbol{Fm} \to \mathcal{P}_i(X)$ is a valuation on $\mathcal{F}$ iff it is a homomorphism between $\boldsymbol{Fm}$ and $\boldsymbol{A}(\mathcal{F})$. Consequently

$$\mathcal{F} \vDash \varphi \text{ iff } \boldsymbol{A}(\mathcal{F}) \vDash \varphi \approx 1,$$

for every formula φ.

Lemma 4.3. *Let $\mathcal{F}$ be a frame. Then for each $x \in X$, the set $X - R_\neg(x) = R_\neg(x)^c$ is increasing.*

Proof. Let $x, y, z \in X$, such that $z \leq y$ and $z \in R_\neg(x)^c$. If $y \in R_\neg(x)$, there exists $w \in X$ such that $(x, w) \in R$ and $y \leq w$. As $z \leq y \leq w$ we have

$z \in R_{\neg}(x)$, which is a contradiction. Thus, $R_{\neg}(x)^c$ is an increasing subset of X. □

For sequents we can give two notions of validity. These notions are similar to the notions of global and local validity in Kripke frames for sequents in modal logic (see for instance [4]).

Definition 4.4. Let $\Gamma \vdash \varphi$ be a sequent and let $\mathcal{F}$ be a $\neg$-frame. Then:

1. $\Gamma \vdash \varphi$ is *locally valid* in $\mathcal{F}$ if and only if $\bigcap \{V(\psi) : \psi \in \Gamma\} \subseteq V(\varphi)$ for each valuation V based on $\mathcal{F}$, in symbols $\mathcal{F} \models_l \Gamma \vdash \varphi$.
2. $\Gamma \vdash \varphi$ is *globally valid* in $\mathcal{F}$ if and only if $\bigcap \{V(\psi) : \psi \in \Gamma\} \neq X$ or $V(\varphi) = X$ for each valuation V based on $\mathcal{F}$, in symbols $\mathcal{F} \models_g \Gamma \vdash \varphi$.

We note that

$$\mathcal{F} \models_l \{\varphi_1, \ldots, \varphi_n\} \vdash \varphi \text{ iff } \boldsymbol{A}(\mathcal{F}) \vDash \varphi_1 \wedge \ldots \wedge \varphi_n \wedge \varphi \approx \varphi_1 \wedge \ldots \wedge \varphi_n,$$

and

$$\mathcal{F} \models_g \{\varphi_1, \ldots, \varphi_n\} \vdash \varphi \quad \text{iff} \quad A(\mathcal{F}) \vDash \varphi_1 \approx 1 \,\&\ldots\&\, A(\mathcal{F}) \vDash \varphi_n \approx 1, \text{ implies that } A(\mathcal{F}) \vDash \varphi \approx 1.$$

Let F be a class of frames. We denote by $\vDash_{l(\mathsf{F})}$ the consequence relation defined by $\Gamma \vDash_{l(\mathsf{F})} \varphi$ iff the sequent $\Gamma \vdash \varphi$ is locally valid in every $\neg$-frame $\mathcal{F}$ in F. Similarly, we define the consequence $\vDash_{g(\mathsf{F})}$ as $\Gamma \vDash_{g(\mathsf{F})} \varphi$ iff the sequent $\Gamma \vdash \varphi$ is globally valid in every $\neg$-frame $\mathcal{F}$ in F. We note that if $\Gamma \vdash \varphi$ is locally valid in $\mathcal{F}$, then $\Gamma \vdash \varphi$ is globally valid in $\mathcal{F}$.

Let be $\mathcal{S}$ any deductive system that is an extension of the deductive system $\mathcal{S}(\mathsf{N})$. We will denote by $\mathrm{Fr}(\mathcal{S})$ the class of all frames where every sequent of $\mathcal{S}$ is locally valid, i.e., $\mathrm{Fr}(\mathcal{S}) = \{\mathcal{F} \mid \mathcal{F} \models_l \Gamma \vdash \varphi, \text{ for any } \Gamma \vdash_{\mathcal{S}} \varphi\}$. A deductive system $\mathcal{S}$ is *characterized* by a class F of frames or *it is complete relative to a class* F of frames, when $\Gamma \vdash \varphi \in \mathcal{S}$ iff $\mathcal{F} \models_l \Gamma \vdash \varphi$, for any $\mathcal{F} \in \mathsf{F}$. Moreover, it is *frame complete* when $\Gamma \vdash \varphi \in \mathcal{S}$ iff $\mathcal{F} \models_l \Gamma \vdash \varphi$, for any $\mathcal{F} \in \mathrm{Fr}(\mathcal{S})$. It is clear that a deductive system $\mathcal{S}$ is frame complete if and only if it is characterized by some class of frames.

Let $\boldsymbol{A}$ be a $\neg$-lattice. The set of all prime filters of $\boldsymbol{A}$ is denoted by $X(A)$. The $\neg$-frame of $\boldsymbol{A}$ is the structure $\mathcal{F}(\boldsymbol{A}) = \langle X(A), \subseteq, R \rangle$, where the relation $R \subseteq X(A) \times X(A)$ is given by:

$$(x, y) \in R \text{ iff } \neg^{-1}(x) \cap y = \emptyset,$$

where $\neg^{-1}(x) = \{a \in A : \neg a \in x\}$ (see [5] or [6] for more details).

Lemma 4.5. [5] *Let* $\boldsymbol{A} \in \mathsf{N}$. *Let* $x \in X(A)$. *Then for each* $a \in A$, $\neg a \notin x$ *iff there is* $y \in X(A)$ *such that* $(x, y) \in R$ *and* $a \in y$.

Lemma 4.6. [5] *Let* $\boldsymbol{A} \in \mathsf{N}$. *Then* $\mathcal{F}(\boldsymbol{A})$ *is a* $\neg$*-frame and the mapping* $\sigma : A \to \mathcal{P}_i(X(A))$ *is a one to one homomorphism form* $\boldsymbol{A}$ *into* $\boldsymbol{A}(\mathcal{F}(\boldsymbol{A})) = \left\langle \mathcal{P}_i(X(A)), \neg_{R_\neg} \right\rangle$, *where* $\sigma(a) = \{P \in X(A) \mid a \in P\}$.

From Lemma 4.6 we have that $\boldsymbol{A}(\mathcal{F}(\boldsymbol{A})) = \langle \mathcal{P}_i(X(A)), \neg_R \rangle$ is a $\neg$-lattice, called the *canonical extension* of $\boldsymbol{A}$.

Proposition 4.7 (Soundness)**.** *Let* V *be a variety of* $\neg$*-lattices. Let* $\Gamma \cup \{\varphi\} \subseteq Fm$.

1. *If* $\Gamma \vdash_{\mathcal{S}(\mathsf{V})} \varphi$ *then* $\Gamma \vDash_{l(\mathrm{Fr}(\mathcal{S}(\mathsf{V})))} \varphi$.
2. *If* $\Gamma \vdash_{\mathcal{S}(\mathsf{V},1)} \varphi$ *then* $\Gamma \vDash_{g(\mathrm{Fr}(\mathcal{S}(\mathsf{V})))} \varphi$.

Proof. (1) Suppose that $\Gamma \vdash_{\mathcal{S}(\mathsf{V})} \varphi$. Let $\mathcal{F}$ be a $\neg$-frame of $\mathcal{S}(\mathsf{V})$, i.e. $\mathcal{F} \in \mathrm{Fr}(\mathcal{S}(\mathsf{V}))$. Let V be a valuation based on $\mathcal{F}$. Let $x \in V(\psi)$, for all $\psi \in \Gamma$. As before we have V is a homomorphism from $\boldsymbol{Fm}$ into the $\neg$-lattice $\boldsymbol{A}(\mathcal{F})$. Since $\mathcal{F}$ is a $\neg$-frame of $\mathcal{S}(\mathsf{V})$, we have $\boldsymbol{A}(\mathcal{F}) \in \mathsf{V}$. Thus by definition of $\mathcal{S}(\mathsf{V})$, there exist $\psi_1, \ldots, \psi_n \in \Gamma$ such that $V(\psi_1 \wedge \ldots \wedge \psi_n) = V(\psi_1) \cap \ldots \cap V(\psi_n) \subseteq V(\varphi)$. Then $x \in V(\varphi)$. We conclude that $\Gamma \vDash_{l(\mathrm{Fr}(\mathcal{S}(\mathsf{V})))} \varphi$.

(2) Suppose that $\Gamma \vdash_{\mathcal{S}(\mathsf{V},1)} \varphi$. Let $\mathcal{F}$ be a $\neg$-frame of $\mathcal{S}(\mathsf{V})$, i.e. $\mathcal{F} \in \mathrm{Fr}(\mathcal{S}(\mathsf{V}))$. Let V a valuation based on $\mathcal{F}$ such that $V(\psi) = X$ for each $\psi \in \Gamma$. Consider the $\neg$-lattice $\boldsymbol{A}(\mathcal{F})$. Since V is a homomorphism from $\boldsymbol{Fm}$ into $\boldsymbol{A}(\mathcal{F})$, we get by definition of $\vdash_{\mathcal{S}(\mathsf{WN},1)}$ that $V(\varphi) = X$. Thus we conclude that $\mathcal{F} \models_{g(\mathrm{Fr}(\mathcal{S}(\mathsf{V})))} \Gamma \vdash \varphi$. □

Definition 4.8. Let V be a variety of $\neg$-lattices. We shall say that V is *canonical* if $\boldsymbol{A}(\mathcal{F}(\boldsymbol{A})) \in \mathsf{V}$ whenever $A \in \mathsf{V}$.

The notion of canonical varieties is the algebraic counterpart of canonical modal logics (see [4]). Now we are ready to prove one of the main results of this paper.

Theorem 4.9. *Let* V *be a variety of* $\neg$*-lattices. If* V *is canonical, then* $\Gamma \vDash_{g(\mathrm{Fr}(\mathcal{S}(\mathsf{V})))} \varphi$ *implies that* $\Gamma \vdash_{\mathcal{S}(\mathsf{V},1)} \varphi$.

Proof. Suppose that $\Gamma \vDash_{g(\mathsf{F})} \varphi$. Let $\boldsymbol{A}$ be an algebra, and $h : \boldsymbol{Fm} \to \boldsymbol{A}$ a homomorphism such that $h(\psi) = 1$ for each $\psi \in \Gamma$. As $\sigma : \boldsymbol{A} \to \boldsymbol{A}(\mathcal{F}(\boldsymbol{A}))$ is an one to one homomorphism, the composition $\sigma \circ h$ is a homomorphism from $\boldsymbol{Fm}$ into $\boldsymbol{A}(\mathcal{F}(\boldsymbol{A}))$, i.e., $\sigma \circ h$ is a valuation based on the $\neg$-frame $\mathcal{F}(\boldsymbol{A}) = \langle X(A), \subseteq, R\rangle$. By hypothesis $(\sigma \circ h)(\varphi) = \sigma(h(\varphi)) = X(A) = \sigma(1)$. As σ is injective, $h(\varphi) = 1$. Thus, $\Gamma \vdash_{\mathcal{S}(\mathsf{N},1)} \varphi$. □

Let $\boldsymbol{A}$ be a $\neg$-lattice . The filter generated by a set $H \subset A$ is denoted by $F(H)$.

Theorem 4.10. *Let* V *be a variety of* $\neg$*-lattices. If* V *is canonical, then* $\Gamma \vDash_{l(\mathrm{Fr}(\mathcal{S}(\mathsf{V})))} \varphi$ *implies that* $\Gamma \vdash_{\mathcal{S}(\mathsf{V})} \varphi$.

Proof. Assume that $\Gamma \vDash_{l(\mathrm{Fr}(\mathcal{S}(\mathsf{V})))} \varphi$. But suppose that $\Gamma \nvdash_{\mathcal{S}(\mathsf{V})} \varphi$. Then there exists $\boldsymbol{A} \in \mathsf{V}$, and there exists a homomorphism $h : \boldsymbol{Fm} \to \boldsymbol{A}$ such that $h(\varphi) \notin F(\{h(\psi) \mid \psi \in \Gamma\})$. Then there exist a prime filter x of $\boldsymbol{A}$ such that $h(\varphi) \notin x$ and $h(\psi) \in x$, for all $\psi \in \Gamma$. Recall the composition $\sigma \circ h$ is a homomorphism from $\boldsymbol{Fm}$ into $\boldsymbol{A}(\mathcal{F}(\boldsymbol{A}))$. As V is canonical, we get $\boldsymbol{A}(\mathcal{F}(\boldsymbol{A})) \in \mathsf{V}$. So $\sigma \circ h$ is a valuation based in a $\neg$-frame $\mathcal{F}(\boldsymbol{A}) = \langle X(A), \subseteq, R\rangle$ of $\mathrm{Fr}(\mathcal{S}(\mathsf{V}))$. Since, $x \in (\sigma \circ h)(\psi) = \sigma(h(\psi))$ for each $\psi \in \Gamma$, we get $x \in (\sigma \circ h)(\varphi) = \sigma(h(\varphi))$, i.e., $\bigcap\{(\sigma \circ h)(\psi) : \psi \in \Gamma\} \nsubseteq (\sigma \circ h)(\varphi)$, which is a contradiction. Thus, $h(\varphi) \in F(\{h(\psi) \mid \psi \in \Gamma\})$. □

As consequence of Theorem 3.1 and Theorem 4.10 we have that

$$\vdash_{\mathcal{S}_\neg} = \vdash_{\mathcal{S}(\mathsf{N})} = \vDash_{l(\mathsf{F})},$$

where F is the class of all $\neg$-frames.

5. Correspondence results

In this section we will characterize the class of $\neg$-frames for some extensions of $\mathcal{S}_\neg$ considered in section 3. Let $\mathcal{F} = \langle X, \leq, R\rangle$ be a $\neg$-frame. Recall that $R_\neg$ is the composition $R \circ \leq^{-1}$.

Theorem 5.1. *Let* $\mathcal{F}$ *be a* $\neg$*-frame. Then*

1. *The rule* (**ANT**) $\dfrac{\varphi, \psi \vdash \alpha}{\varphi, \neg\alpha \vdash \neg\psi}$ *is valid in* $\mathcal{F}$ *iff* $\forall xy(xRy \Rightarrow [x) \cap [y) \cap R_\neg(x) \neq \emptyset)$.

2. The rule (**NE**) $\dfrac{\varphi \vdash \neg\psi}{\varphi, \psi \vdash \bot}$ *is valid in* $\mathcal{F}$ *iff* $\forall x\ ([x) \subseteq R_\neg(x))$.

3. The rule (**NI**) $\dfrac{\varphi \vdash \alpha}{\varphi, \neg\alpha \vdash \bot}$ *is valid in* $\mathcal{F}$ *iff* $\forall xy(xRy \Rightarrow [x) \cap [y) \neq \emptyset)$.

Proof. (1) $\Rightarrow$) Let $x, y \in X$ such that $(x, y) \in R$. Suppose that $[x) \cap [y) \cap R_\neg(x) = \emptyset$. Let V be the valuation defined by

$$V(p) = [x)\,,\ V(q) = [y)\,, \text{ and } V(r) = R_\neg(x)^c.$$

Then $V(p) \cap V(q) = V(p \wedge q) \subseteq V(r)$. So by the assumption, $V(p) \cap V(\neg r) \subseteq V(\neg q)$. Since $x \in V(p)$, and $R(x) \cap R_\neg(x)^c = \emptyset$, we get that $x \in V(p) \cap V(\neg r) \subseteq V(\neg q)$. Then $R(x) \cap [y) = \emptyset$, which is a contradiction because $y \in R(x)$. Thus $[x) \cap [y) \cap \bar{R}(x) \neq \emptyset$.

$\Leftarrow$) Let $\varphi, \psi \in Fm$ be such that $\varphi, \psi \vdash \alpha$. Then $V(\varphi) \cap V(\psi) \subseteq V(\alpha)$. We prove that $V(\varphi) \cap V(\neg\alpha) \subseteq V(\neg\psi)$. Let $x \in V(\varphi) \cap V(\neg\alpha)$. Suppose that $x \notin V(\neg\psi) = \neg_{R_\neg}(V(\psi))$. Then there exists $y \in R_\neg(x)$ and $y \in V(\psi)$. As $[x) \cap [y) \cap R_\neg(x) \neq \emptyset$, there exists $z \in X$ such that $x \leq z$, $y \leq z$, and $(x, z) \in R_\neg$. As $V(\varphi)$ and $V(\psi)$ are increasing sets, $z \in V(\varphi) \cap V(\psi)$. So, $z \in V(\alpha)$. Then $z \in V(\alpha) \cap R_\neg(x)$ which is impossible, because $x \in V(\neg\alpha)$. Thus, $V(\varphi) \cap V(\neg\alpha) \subseteq V(\neg\psi)$.

(2) $\Rightarrow$) Consider the valuation V defined in $\mathcal{F}$ by

$$V(p) = [x) \text{ and } V(q) = R^c_\neg(x).$$

From Lemma 4.3 we get that V is well-defined. We prove that $V(p) \subseteq V(\neg q)$. Let $z \in [x)$. As $x \leq z$, $R(z) \subseteq R_\neg(x)$. So, $R(z) \cap R^c_\neg(x) = \emptyset$, i.e. $z \in V(\neg q)$. Thus, $V(p) \cap V(q) = [x) \cap R^c_\neg(x) = \emptyset$, i.e. $[x) \subseteq R_\neg(x)$.

$\Leftarrow$) Suppose that $V(\varphi) \subseteq V(\neg\psi)$. If there exists $x \in V(\varphi) \cap V(\psi)$, then $x \in V(\neg\psi)$, i.e., $x \in N$ and $R_\neg(x) \cap V(\psi) = \emptyset$. By hypothesis $[x) \subseteq R_\neg(x)$. Then, $x \in R_\neg(x) \cap V(\psi) = \emptyset$, which is a contradiction.

The proof of (3) is similar to the proof of (1). □

Corollary 5.2. *Let* $\mathcal{F}$ *be a* $\neg$*-frame. Then* $\varphi \wedge \neg(\varphi \wedge \psi) \vdash \neg\psi$ *is valid in* $\mathcal{F}$ *iff* $\forall xy(xRy \Rightarrow [x) \cap [y) \cap R_\neg(x) \neq \emptyset)$.

Proof. It follow by Lemma 3.2 and item (1) of Theorem 5.1. □

Corollary 5.3. *Let $\mathcal{F}$ be a frame. If the rules* (**NI**) *and* (**NE**) *are valid in $\mathcal{F}$, then*

$$\forall x \forall y \ ([x) \cap [y) \neq \emptyset \leftrightarrow x R_{\neg} y),$$

i.e. $R_{\neg}$ is definable by $\leq$.

Proof. $\Rightarrow$) Let $x, y \in X$ such that $[x) \cap [y) \neq \emptyset$. Then there exists $z \in X$ such that $x \leq z$ and $y \leq z$. By point 2 of Theorem 5.1, $z \in R_{\neg}(x)$. So, there exists $w \in X$ such that $(x, w) \in R$ and $z \leq w$. As $y \leq z \leq w$, we get $(x, y) \in R_{\neg}$.

$\Leftarrow$) Let $x, y \in X$ such that $(x, y) \in R_{\neg}$. By point (3) of Theorem 5.1 we get $[x) \cap [y) \neq \emptyset$. □

By the previous corollary we have that the class of $\neg$-frames of the deductive system $\mathcal{S}(\mathsf{PA})$ is the class of all posets, because the binary relation $R_{\neg}$ is defined by the order $\leq$.

Theorem 5.4. *Let $\mathcal{F}$ be a $\neg$-frame.*

1. $\top \vdash \neg(\varphi \wedge \neg\varphi)$ *is valid in $\mathcal{F}$ iff* $\forall x \forall y \in X (x R y \Rightarrow y R_{\neg} y)$.
2. $\varphi \vdash \neg\neg\varphi$ *is valid in $\mathcal{F}$ iff $R_{\neg}$ is symmetrical.*
3. $\vdash \neg\varphi \vee \neg\neg\varphi$ *is valid in $\mathcal{F}$ iff* $R_{\neg} \circ R_{\neg}^{-1} \subseteq R_{\neg}$ *($R_{\neg}$ is euclidean)*
4. $\varphi \wedge \neg\varphi \vdash \bot$ *is valid in $\mathcal{F}$ iff $R_{\neg}$ is reflexive.*
5. $\varphi \wedge \neg\varphi \vdash \neg\psi$ *is valid in $\mathcal{F}$ iff* $\forall x \forall y \in X (x R y \Rightarrow x R_{\neg} x)$.
6. $\neg\top \vdash \bot$ *is valid in $\mathcal{F}$ iff* $\forall x \exists y \, (x R_{\neg} y)$ *($R_{\neg}$ is serial).*

Proof. (1) $\Rightarrow$) Let $x, y \in X$ be such that $(x, y) \in R$. Suppose that $(y, y) \notin R_{\neg}$, i.e. $y \notin R_{\neg}(y)$. Let us consider the valuation V defined by

$$V(p) = R_{\neg}(y)^c.$$

So, $y \in R(x) \cap V(p)$. Since by the assumption $x \in V(\neg(p \wedge \neg p)) = X$,

$$R(x) \cap V(p) \cap V(\neg p) = \emptyset.$$

Since $y \in R(x) \cap V(p)$, we get $y \notin V(\neg p)$. It follows that

$$R(y) \cap V(p) = R(x) \cap R_{\neg}(y)^c \neq \emptyset,$$

which is a contradiction. Thus, $y \in R_{\neg}(y)$.

$\Leftarrow$) Suppose that $\mathcal{F} \nvDash \top \vdash \neg(\varphi \wedge \neg\varphi)$, for some formula φ. Then there exists a valuation V on $\mathcal{F}$ such that $V(\neg(\varphi \wedge \neg\varphi)) \neq X$. Then there exists $x \in X$ such that $R(x) \cap V(\varphi) \cap V(\neg\varphi) \neq \emptyset$. So there exists $y \in R(x)$, $y \in V(\varphi)$, and $R(y) \cap V(\varphi) = R_{\neg}(y) \cap V(\varphi) = \emptyset$. But $y \in R_{\neg}(y)$, and as $y \in V(\varphi)$, we get $y \in R_{\neg}(y) \cap V(\varphi)$, which is impossible. Thus, $V(\neg(\varphi \wedge \neg\varphi)) = X$.

(2) $\Rightarrow$) Let $x, y \in X$ be such that $(x, y) \in R_{\neg}$. Suppose that $x \notin R_{\neg}(y)$, i.e. $x \in R_{\neg}(y)^c$. From Lemma 4.3 we can consider the valuation $V(p) = R(y)^c$. As $x \in R_{\neg}(y)^c = V(p) \subseteq V(\neg\neg p)$, $R_{\neg}(x) \cap V(\neg p) = \emptyset$. Then $y \notin V(\neg p)$, i.e. $R_{\neg}(y) \cap V(p) = R_{\neg}(y) \cap R_{\neg}(y)^c \neq \emptyset$, which is a contradiction. Thus, $x \in R(y)$.

$\Leftarrow$) Let φ be a formula. Let $x \in V(\varphi)$. We need to prove that $R_{\neg}(x) \cap V(\neg\varphi) = \emptyset$. If there exists $y \in R_{\neg}(x) \cap V(\neg\varphi)$, we get $R_{\neg}(y) \cap V(\varphi) = \emptyset$, but as $R_{\neg}$ is symmetrical, $x \in R_{\neg}(y)$, and since $x \in V(\varphi)$, we deduce that $R_{\neg}(y) \cap V(\varphi) \neq \emptyset$, which is impossible.

(3) $\Rightarrow$) Let $x, y, z \in X$ be such that $(x, y) \in R_{\neg}$ and $(x, z) \in R_{\neg}$. Suppose that $z \notin R_{\neg}(y)$. Consider the valuation V defined by $V(p) = R_{\neg}(y)^c$. Then $z \in R_{\neg}(x) \cap V(p)$. It follows that $x \notin V(\neg p)$. Since by the assumption $V(\neg p \vee \neg\neg p) = X$, we have that $x \in V(\neg\neg p)$, i.e. $R_{\neg}(x) \cap V(\neg p) = \emptyset$. But as $R_{\neg}(y) \cap R_{\neg}(y)^c = \emptyset$, $y \in V(\neg p)$, and consequently $y \in V(\neg p) \cap R_{\neg}(x)$, which is impossible. Therefore $z \in R_{\neg}(y)$.

$\Leftarrow$) It is easy.

(4) $\Rightarrow$) Suppose that $x \notin R_{\neg}(x)$. Consider the valuation V defined by $V(p) = R_{\neg}(x)^c$. As form the assumption follows that $V(p \wedge \neg p) = V(p) \cap V(\neg p) = \emptyset$, we get that $x \notin V(\neg p)$, i.e., $R_{\neg}(x) \cap V(p) = R_{\neg}(x) \cap R_{\neg}(x)^c \neq \emptyset$, which is an absurd. Thus $R_{\neg}$ is serial.

$\Leftarrow$) It is easy.

(5) $\Rightarrow$) Let $x, y \in X$ be such that $(x, y) \in R$. Suppose that $(x, x) \notin R_{\neg}$. Consider the valuation V defined by $V(p) = R_{\neg}(x)^c$. Then $x \in V(p)$, and as $R_{\neg}(x)^c \cap V(p) = \emptyset$, $x \in V(\neg p)$. Then $x \in V(p) \cap V(\neg p) = V(p \wedge \neg p) \subseteq V(\neg\top)$. So, $R(x) \cap V(\top) = R(x) \cap X = \emptyset$. But this implies that $R(x) = \emptyset$, which is impossible because $y \in R(x)$. Thus $(x, x) \in R_{\neg}$.

$\Leftarrow$) It is easy.

(6) It is easy. □

Theorem 5.5. *If the sequents $\vdash \neg\varphi \vee \neg\neg\varphi$ and $\neg\top \vdash \bot$ are valid in a*

$\neg$-frame $\mathcal{F}$, then $\neg\varphi \wedge \neg\neg\varphi \vdash \bot$ is valid in $\mathcal{F}$ iff $R_\neg$ is transitive.

Proof. $\Rightarrow$) Assume that $\neg\varphi \wedge \neg\neg\varphi \vdash \bot$ is valid in $\mathcal{F}$. Let $x, y, z \in X$ such that $(x, y) \in \forall x \forall y \in X$(if $(x, y) \in R$, then $(y, y) \in R_\neg$). and $(y, z) \in R_\neg$. Suppose that $(x, z) \notin R_\neg$. Consider the valuation $V(p) = R_\neg(x)^c$. Then $y \notin V(\neg p)$. As $\vdash \neg\varphi \vee \neg\neg\varphi$ is valid in $\mathcal{F}$, $V(\neg\varphi) \cup V(\neg\neg\varphi) = X$. So, $y \in V(\neg\neg\varphi)$. As $R_\neg(x) \cap R_\neg(x)^c = \emptyset$, $x \in V(\neg p)$, and since $V(\neg\varphi) \cap V(\neg\neg\varphi) = \emptyset$, we get $x \notin V(\neg\neg\varphi)$. So, $x \in V(\neg\neg\neg\varphi)$, because $V(\neg\neg\varphi) \cup V(\neg\neg\neg\varphi) = X$. As $(x, y) \in R_\neg$, $y \notin V(\neg\neg\varphi)$, which is a contradiction. Thus, $\bar{R}$ is transitive.

$\Leftarrow$) Suppose that there exists a formula φ such that the sequent $\neg\varphi \wedge \neg\neg\varphi \vdash \bot$ is not valid in $\mathcal{F}$. Then there exists a valuation V on $\mathcal{F}$ such that $V(\neg\varphi \wedge \neg\neg\varphi) \neq \emptyset$. Then there exists $x \in V(\neg\varphi) \cap V(\neg\neg\varphi)$. Since $\neg\top \vdash \bot$ is valid in frame $\mathcal{F}$, $R_\neg$ is serial. Thus, there exists $y \in X$ such that $(x, y) \in R_\neg$. Then $y \notin V(\varphi)$ and $y \notin V(\neg\varphi)$. So there exists $z \in X$ such that $(y, z) \in R_\neg$ and $z \in V(\varphi)$. Since $R_\neg$ is transitive, $(x, z) \in R_\neg$, and as $x \in V(\neg\varphi)$,we get $z \notin V(\varphi)$, which is an absurd. Thus, $\neg\varphi \wedge \neg\neg\varphi \vdash \bot$ is valid in $\mathcal{F}$. □

The above results allow us to characterize the class of frames of some extensions of the deductive system $\mathcal{S}(\mathsf{N})$. Recall that $\mathrm{Fr}(\mathcal{S}(\mathsf{V}))$ denotes the class of all $\neg$-frames that satisfies all the sequent of $\mathcal{S}(\mathsf{V})$.

The class $\mathrm{Fr}(\mathcal{S}(\mathsf{WA})))$ is the class of $\neg$-frames $\mathcal{F} = \langle X, \leq, R\rangle$ such that satisfy the following first-order conditions:

$$\forall x \forall y (xRy \Rightarrow [x) \cap [y) \cap R_\neg(x) \neq \emptyset).$$
$$\forall x \forall y (xR_\neg y \Rightarrow yR_\neg x).$$

The class $\mathrm{Fr}(\mathcal{S}(\mathsf{QS})$ is the class of $\neg$-frames where $R_\neg$ is an equivalence; the class of $\neg$-frames of $\mathcal{S}(\mathsf{WQS})$ is the class of$\neg$-frames where $R_\neg$ is serial, transitive and euclidean and, as noted earlier, the class of $\neg$-frames of $\mathcal{S}(\mathsf{PA})$ is the class of all posets.

6. Completeness of extensions of $\mathcal{S}(\mathsf{N})$

To prove that some of the extensions of $\mathcal{S}_\neg$ are complete with respect to their frames we will use the representation theory developed in Section 4 for $\neg$-lattices. First we note that in the $\neg$-frame $\mathcal{F}(A) = \langle X(A), \subseteq, R\rangle$ of a $\neg$-lattice $\boldsymbol{A}$ the relation R satisfies the condition $R = R\circ \subseteq^{-1}$, i.e., $R = R_{\neg}$.

Theorem 6.1. *Let $\boldsymbol{A} \in \mathsf{N}$ and let $\mathcal{F}(A) = \langle X(A), \subseteq, R\rangle$ be its $\neg$-frame.*

1. $\neg(a \wedge \neg a) = 1$ *is valid in* $\boldsymbol{A}$ *iff* $\forall x \forall y (xRy \Rightarrow yRy)$.
2. $a \wedge \neg(a \wedge b) \leq \neg b$ *is valid in* $\boldsymbol{A}$ *iff* $\forall x \forall y (xRy \Rightarrow \exists z \in X(A)\ (x \subseteq z\ \&\ y \subseteq z\ \&\ xRz)$.
3. $a \wedge \neg a \leq \neg b$ *is valid in* $\boldsymbol{A}$ *iff* $\forall x \forall y (xRy \Rightarrow xRx)$.
4. $a \leq \neg\neg a$ *is valid in* $\boldsymbol{A}$ *iff* R *is symmetrical.*
5. $\neg a \vee \neg\neg a = 1$ *is valid in* $\boldsymbol{A}$ *iff* $R \circ R^{-1} \subseteq R$.
6. $\neg 1 = 0$ *iff* R *is serial, i.e.* $R(x) \neq \emptyset$ *for any* $x \in X(A)$.

Proof. (1) Let $x, y \in X(A)$ be such that $(x, y) \in R$. Suppose that $(y, y) \notin R$. Then there exists $a \in A$ such that $\neg a \in y$ and $a \in y$. As y is a filter, $\neg a \wedge a \in y$. Then $\neg(a \wedge \neg a) \notin x$, which is a contradiction, because $1 \in x$.

Suppose that there exists $a \in A$ such that $\neg(a \wedge \neg a) \neq 1$. Then there exists a prime filter x of $\boldsymbol{A}$ such that $\neg(a \wedge \neg a) \notin x$. By Lemma 4.5 there exists $y \in X(A)$ such that $(x, y) \in R$ and $a \wedge \neg a \in y$. As $(y, y) \in R$, and $a \in y$, we have that $\neg a \notin y$, which is a contradiction. Thus, $\neg(a \wedge \neg a) = 1$ is valid in $\boldsymbol{A}$.

(2) Let $x, y \in X(A)$ such that $(x, y) \in R$. Let F be the filter generated by the set $x \cup y$. Since $(x, y) \in R_{\neg}$, $\neg 1 \notin P$. So, $\neg^{-1}(x) \neq A$. We prove that $F \cap \neg^{-1}(x) = \emptyset$. If there exist $a \in x$, $b \in y$ and $c \in \neg^{-1}(x)$ such that $a \wedge b \leq c$, then $\neg c \leq \neg(a \wedge b)$. So $\neg(a \wedge b) \in x$, and as $a \in x$, and $a \wedge \neg(a \wedge b) \leq \neg b$, we get that $\neg b \in x$, but this implies that $b \notin y$, which is an absurd. Thus $F \cap \neg^{-1}(x) = \emptyset$. By the Prime filter theorem, there exists $z \in X(A)$ such that $x \subseteq z$, $y \subseteq z$ and $(x, z) \in R_{\neg}$.

Suppose that there exist $a, b \in A$ such that $a \wedge \neg(a \wedge b) \nleq \neg b$. Then there exists $x \in X(A)$ such that $a \wedge \neg(a \wedge b) \in x$ and $\neg b \notin x$. So, there exists $y \in X(A)$ such that $(x, y) \in R$ and $b \in y$. Then there exists $z \in X(A)$ such that $x \subseteq z$, $y \subseteq z$, and xRz. So, $a, b \in z$, but as $\neg(a \wedge b) \in x$, we have that $a \wedge b \notin z$, which is impossible. Thus, $a \wedge \neg(a \wedge b) \leq \neg b$ is valid in $\boldsymbol{A}$.

(3) It is easy and left to the reader, and the items (4) to (6) were proved in [7]. □

Proposition 6.2. *Let $\boldsymbol{A} \in \mathsf{N}$ and let $\mathcal{F}(A) = \langle X(A), \subseteq, R\rangle$ be its ¬-frame. Then the following conditions are equivalent:*

1. $a \wedge \neg a = 0$, *is valid in* $\boldsymbol{A}$.
2. *If* $a \leq \neg b$, *then* $a \wedge b = 0$, *for all* $a, b \in A$.
3. R *is reflexive.*
4. $\forall x \forall y(x \subseteq y$, *implies that* $(x, y) \in R)$.

Proof. (1) ⇒ (2) Let $a, b \in A$ such that $a \wedge \neg b = a$. Then, $a \wedge \neg b \wedge b = a \wedge b = 0$.

(2) ⇒ (3) Let $x \in X(A)$. Let $\neg a \in x$. As $\neg a \leq \neg a$, $a \wedge \neg a = 0$. Thus, $a \notin x$.

(3) ⇒ (4)Let $x, y \in X(A)$. Suppose that $x \subseteq y$. Let $\neg a \in x$. Then $\neg a \in y$, and as $R_\neg$ is reflexive, $a \notin y$. Thus, $(x, y) \in R_\neg$.

(4) ⇒ (1) If $\neg a \wedge a \neq 0$, there exists $x \in X(\boldsymbol{A})$ such that $\neg a \wedge a \in x$. Since $x \subseteq x$, $(x, x) \in R_\neg$, which is a contradiction. Thus, $\neg a \wedge a = 0$. □

Proposition 6.3. *Let $\boldsymbol{A} \in \mathsf{N}$ and let $\mathcal{F}(A) = \langle X(A), \subseteq, R\rangle$ be its ¬-frame. Then the following conditions are equivalent:*

1. *If* $a \wedge b = 0$, *then* $a \leq \neg b$, *for all* $a, b \in A$.
2. $\forall x \forall y(xRy \Rightarrow \exists z(x \subseteq z$ *and* $y \subseteq z)$.

Proof. (1) ⇒ (2). Let $x, y \in X(A)$ such that $(x, y) \in R_\neg$. Let $F(x \cup y)$ be the filter generated by $x \cup y$. If $0 \in F(x \cup y)$, there exists $a \in x$, and $b \in y$ such that $a \wedge b = 0$. Then $a \leq \neg b$, and thus $\neg b \in x$, but as $(x, y) \in R$, $b \notin y$, which is a contradiction.

(2) ⇒ (1). Suppose that there exists $a, b \in A$ such that $a \wedge b = 0$ but $a \nleq \neg b$. Then $a \in x$ and $\neg b \notin x$, for some prime filter x. So, there exists $y \in X(A)$ such that $(x, y) \in R$ and $b \in y$. By hypothesis there exists $z \in X(A)$ such that $x \subseteq z$ and $y \subseteq z$. So, $a \wedge b = 0 \in z$, which is impossible. □

Corollary 6.4. *[5] Let $\boldsymbol{A} \in \mathsf{N}$. Then $\boldsymbol{A}$ is a bounded distributive pseudocomplemented iff $\forall x \forall y((x, y) \in R$ iff $F(x \cup y) \neq A)$.*

We note that from the previous results, if $\boldsymbol{A}$ belong to some of the varieties WA, QS, WQS, or PA, then $\boldsymbol{A}(\mathcal{F}(\boldsymbol{A}))$ belongs to the same variety. Thus, we have that the varieties WA, QS, WQS, and PA, are canonical. As consequence we have the following result.

Theorem 6.5. *The varieties N, WA, QS, WQS, and PA are canonical, and the deductive systems $\mathcal{S}(\mathsf{N})$, $\mathcal{S}(\mathsf{WA})$, $\mathcal{S}(\mathsf{QS})$, $\mathcal{S}(\mathsf{WQS})$, and $\mathcal{S}(\mathsf{PA})$ are frame complete.*

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