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José L. CASTIGLIONI , Marta S. SAGASTUME
and Hernán J. SAN MARTÍN

ON FRONTAL HEYTING ALGEBRAS

A b s t r a c t. A frontal operator in a Heyting algebra is an expansive operator preserving finite meets which also satisfies the equation $\tau(x) \leq y \vee (y \to x)$. A frontal Heyting algebra is a pair (H, τ), where H is a Heyting algebra and τ a frontal operator on H. Frontal operators are always compatible, but not necessarily new or implicit in the sense of Caicedo and Cignoli (*An algebraic approach to intuitionistic connectives.* Journal of Symbolic Logic, 66, N°4 (2001), 1620-1636). Classical examples of new implicit frontal operators are the functions γ, (op. cit., Example 3.1), the successor (op. cit., Example 5.2), and Gabbay's operation (op. cit., Example 5.3).
We study a Priestley duality for the category of frontal Heyting algebras and in particular for the varieties of Heyting algebras with each one of the implicit operations given as examples.
The topological approach of the compatibility of operators seems to be important in the research of affin completeness of Heyting algebras with additional compatible operations. This problem have also a logical point of view. In fact, we look for some complete propositional intuitionistic calculus enriched with implicit connectives.

1. Introduction

We assume that the reader is familiar with the theory of Heyting algebras (see [1]). If we considerer intuitionistic and intermediate propositional calculus as logics with truth values in Heyting algebras, it is natural to consider new connectives for these logics as operations in the algebras. For example, it was considered in [6] the modalized Heyting calculus mHC, which consists of an augmentation of the Heyting propositional calculus by a modal operator. The algebraic models of mHC are Heyting algebras with a unary operator subject to additional identities. These identities must be the algebraic counterpart of the axioms that the modal operator satisfies on the logic.

A frontal Heyting algebra is an algebra $(H, \wedge, \vee, \rightarrow, \tau, 0, 1)$ such that $(H, \wedge, \vee, \rightarrow, 0, 1)$ is a Heyting algebra and τ is a unary operator satisfying the following equations:

(f1) $\tau(x \wedge y) = \tau(x) \wedge \tau(y)$,

(f2) $x \leq \tau(x)$,

(f3) $\tau(x) \leq y \vee (y \rightarrow x)$

The operator τ will be called *frontal*.

We write (H, τ) for short. We say that a Heyting frontal algebra (H, τ) is a Heyting algebra with *successor* if the operator satisfies the additional condition

(f4) $\tau(x) \rightarrow x \leq x$.

The class of frontal Heyting algebras is denoted by **fHA**.

The class **fHA** can be seen as the category whose objects are frontal Heyting algebras and whose morphisms are Heyting morphisms which preserve the frontal operator; these maps are called *frontal Heyting morphisms*.

In [2] it was proved that for any map $h : H \rightarrow H$ in a Heyting algebra H, h is a compatible function of H if and only if $h(x \wedge y) \wedge y = h(x) \wedge y$,

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for all $x, y \in H$. For this reason we have that if (H, τ) is a frontal Heyting algebra then τ is a compatible function of H, by $(f1)$ and $(f2)$.

A set $E(h)$ of equations in the signature of Heyting algebras augmented with the unary function symbol h will be said to define an implicit operation of Heyting algebras if for any Heyting algebra H there is at most one function $h_H : H \to H$. Function h will be an implicit compatible operation provided all h_H are compatible. In case that τ be a frontal implicit operation we call (H, τ) a τ-Heyting algebra.

For each particular τ we will note $\tau\mathbf{HA}$ the full subcategory of **fHA** whose objects are the τ-Heyting algebras.

In section 2 we give a sufficient condition for an operator to be frontal and we study some examples of them: S, γ and G (see [2]). Besides we prove properties about these functions. In section 3 we extend Heyting duality (see [10]) to the category **fHA**. In section 4 we give some applications of the duality developed in section 3. First, we give a characterization for subalgebras of a frontal Heyting algebra. Then we give an easy description of the representation theory of Heyting algebras that admit any of the operators given in section 2.

2. Frontal Heyting algebras

In this section we give a sufficient condition for an operator to be frontal and we study some examples of them. Observe that for every Heyting algebra H there exists a map $\tau : H \to H$ such that the algebra (H, τ) is a frontal Heyting algebra, i.e., the identity.

Proposition 2.1. *Let H be a Heyting algebra and $P : H \times H \to H$ a map that satisfies the following conditions for every $x, y, z \in H$:*

(a) $x \leq P(x, y)$,

(b) $P(x, y) \leq y \vee (y \to x)$,

(c) $P(x \wedge y, z) = P(x, z) \wedge P(y, z)$,

(d) *If* $y \geq z$ *then* $P(x, y) \leq P(x, z)$.

If $\tau : H \to H$ given by $\tau(x) = min\{y \in H : P(x, y) \leq y\}$ defines a function $\tau : H \to H$, then τ is a frontal operator on H.

Proof. By (a) we have that $x \leq P(x, \tau(x)) \leq \tau(x)$, so $x \leq \tau(x)$. By (b) we have that $P(x, y \vee (y \to x)) \leq y \vee (y \to x) \vee ((y \vee (y \to x)) \to x)) = y \vee (y \to x)$, so $\tau(x) \leq y \vee (y \to x)$. By (c) P is monotone in the first coordinate, so if $z \leq w$ then $P(z, \tau(w)) \leq P(w, \tau(w)) \leq \tau(w)$. By this reason $\tau(z) \leq \tau(w)$. Being τ monotone we have that $\tau(x \wedge y) \leq \tau(x) \wedge \tau(y)$. On the other hand, the equations $\tau(x) \leq \tau(x \wedge y) \vee P(x, \tau(x \wedge y))$ and $\tau(y) \leq \tau(x \wedge y) \vee P(y, \tau(x \wedge y))$ hold because by (d) we have that $P(x, a \vee P(x, a)) \leq a \vee P(x, a)$ and $P(x, b \vee P(y, b)) \leq b \vee P(y, b)$, for every $a, b \in H$ (in particular it holds for $a = b = \tau(x \wedge y)$). Then taking $\wedge$ in both members of these inequalities we have, using (c), that $\tau(x) \wedge \tau(y) \leq \tau(x \wedge y)$. Therefore $\tau(x) \wedge \tau(y) = \tau(x \wedge y)$. □

The system $E(S)$ consisting of the following equations given in [2] (see also [9]) defines an implicit compatible operation S of Heyting algebras:

(S1) $x \leq S(x)$,

(S2) $S(x) \leq y \vee (y \to x)$,

(S3) $S(x) \to x = x$.

Equivalently, S can be defined as the unary function

$$S(x) = min\ \{y : y \to x \leq y\}.$$

To prove that, recall that the following fact holds in any Heyting algebra:

$$y \to x \leq y \Leftrightarrow y \to x = x \text{ and } x \leq y. \tag{1}$$

We define the filter $S_x = \{y \in H : y \to x \leq y\}$ and suppose that $S(x)$ satisfies equations $(S1)$, $(S2)$ and $(S3)$. By $(S1)$ and $(S3)$ we have that $S(x) \in S_x$. Let now $y \in S_x$. Then $(S2)$ implies that $S(x) \leq y \vee (y \to x) = y$, so $S(x) = min\ S_x$. Conversely, let $S(x) = min\ S_x$. As $S(x) \in S_x$, by (1) equations $(S1)$ and $(S3)$ hold. Note that $(y \vee (y \to x)) \to x \leq y \vee (y \to x)$, so $S(x) \leq y \vee (y \to x)$. Hence $(S2)$ holds.

Lemma 2.2. *Let H be a Heyting algebra such that the function S exists. Then*

$$S(x \wedge y) = S(x) \wedge S(y),$$

for all $x, y \in H$.

Proof. It is a consequence of Proposition 2.1 and the fact that for S, $P(x, y) = y \to x$. $\square$

Proposition 2.3. *The successor is also implicitly defined by equations* $(f1)$, $(f2)$, $(f3)$ *and* $(S3)$.

Proof. It is a consequence of Lemma 2.2. $\square$

Let H be a Heyting algebra. We write $\neg x$ in place of $x \to 0$.
The system $E(\gamma)$ consisting of the following equations given in [2] defines an implicit compatible operation γ of Heyting algebras:

(γ_1) $\neg\gamma(0) = 0$,

(γ_2) $\gamma(0) \leq (x \vee \neg x)$,

(γ_3) $\gamma(x) = x \vee \gamma(0)$.

In an equivalent way, it is easy to prove that γ can be defined as the unary function

$$\gamma(x) = min\ \{y : \neg y \vee x \leq y\}.$$

Proposition 2.4. *Let H be a Heyting algebra.*
Function γ is also implicitly defined by equations $(f1)$, $(f2)$, $(f3)$ *and the following additional equations:*

(γ_4) $\neg\gamma(0) = 0$,

(γ_5) $\gamma(x) \leq x \vee \gamma(0)$.

Proof. Straightforward. $\square$

The system $E(G)$ consisting of the following equations given in [7] defines an implicit compatible function G of Heyting algebras

(G1) $G(x) \leq y \vee (y \to x)$,

(G2) $x \to y \leq G(x) \to G(y)$,

(G3) $x \leq G(x)$,

(G4) $G(x) \leq \neg\neg x$,

(G5) $G(x) \to x \leq \neg\neg x \to x$.

This function will be called *Gabbay´s function.* It is proven in [12] that $(G2)$ is a consequence of the other equations. In an equivalent way, G can be defined as the unary function

$$G(x) = min\ \{y : (y \to x) \wedge \neg\neg x \leq y\}.$$

To prove this fact recall that in any Heyting algebra H,

$$(y \to x) \wedge \neg\neg x \leq y \Leftrightarrow y \to x \leq \neg\neg x \to x \text{ and } x \leq y \tag{2}$$

Let $G_x = \{y \in H : (y \to x) \wedge \neg\neg x \leq y\}$. Then by (2) we conclude that $G_x = \{y \in H : y \to x \leq \neg\neg x \to x,\ x \leq y\}$. We suppose that G exists. By $(G3)$ and $(G5)$ we have that $G(x) \in G_x$. Let $y \in G_x$. Then by $(G1)$ and $(G4)$ we conclude that $G(x) \leq (y \wedge \neg\neg x) \vee ((y \to x) \wedge \neg\neg x) \leq (y \wedge \neg\neg x) \vee y = y$, so $G(x) = min\ G_x$. Conversely, let $G(x) = min\ G_x$. Then $(G1)$ and $(G5)$ follow from the fact that $G(x) \in G_x$. As $\neg\neg x \in G_x$ we conclude that $(G4)$ holds. As $x \leq y \vee (y \to x)$ and $(y \vee (y \to x)) \to x \leq \neg\neg x \to x$ then $y \vee (y \to x) \in G_x$, so $(G1)$ holds.

Lemma 2.5. *Let H be a Heyting algebra such that the function G exists. Then*

$$G(x \wedge y) = G(x) \wedge G(y),$$

for all $x, y \in H$.

Proof. It is a consequence of Proposition 2.1 and the fact that for G, $P(x, y) = (y \to x) \wedge \neg\neg x$. □

***Remark* 2.6.** Gabbay´s function is implicitly defined by equations $(f1)$, $(f2)$, $(f3)$, $(G4)$ and $(G5)$.

Proposition 2.7. *Let H be a Heyting algebra.*
Gabbay´s function exists if and only if (H, G) is a frontal Heyting algebra and G satifies the additional equation

$$G(x) \to x = \neg\neg x \to x \tag{3}$$

Proof. Let G be the Gabbay´s function. By Lemma 2.5, G is a frontal operator. By $(G4)$ and $(G5)$ we conclude that $G(x) \to x = \neg\neg x \to x$. Conversely, let G a frontal operator which satisfies (3). We only need to prove $(G4)$. We have that $G(\neg\neg x) \to \neg\neg x = 1$, so $G(\neg\neg x) \leq \neg\neg x$. By $(f1)$ we have that G is monotone, so $G(x) \leq G(\neg\neg x) \leq \neg\neg x$. □

Since S, γ and G do not exist in the Heyting algebra $[0,1]$, we get that they are not terms in the vocabulary of Heyting algebras.

Caicedo and Cignoli prove in [2] the following facts: γ and G are definable in terms of S, since $\gamma(x) = x \vee S(0)$ and $G(x) = S(x) \wedge \neg\neg x$. S is not definable from G or γ. G and γ are not mutually definable. However S is definable from G and γ as $S(x) = \gamma(x) \vee G(x)$.

3. Representation theory

We recall that Heyting duality (see [8] or [10]) establishes a dual equivalence between the category **HA** of Heyting algebras and homomorphisms of Heyting algebras and the category **HS** of Heyting spaces and p-continuous morphisms (called Heyting morphisms),

$$\mathcal{PF} : \mathbf{HA} \leftrightarrows \mathbf{HS}^{op} : \mathcal{CU}$$

For every Heyting algebra H, $\mathcal{PF}(H)$ denotes the set of prime filters of H. For every $(X, \leq)$ Heyting space, $\mathcal{CU}(X, \leq)$ denotes the set of clopen upsets of $(X, \leq)$. We have that $\varphi_H(x) = \{P \in \mathcal{PF}(H) : x \in P\}$ is an isomorphism of Heyting algebras between H and $\mathcal{CU}(\mathcal{PF}(H), \subseteq)$ and $G_X(x) = \{U \in \mathcal{CU}(X, \leq) : x \in U\}$ is an isomorphism of Heyting spaces between $(X, \leq)$ and $\mathcal{PF}(\mathcal{CU}(X, \leq), \subseteq)$. Both isomorphisms are natural.

In this section we extend Heyting duality to the category **fHA** and we complete results given in [6] and [11] (section 5). We want to restrict Heyting duality to the category **fHA**. A *Rf-Heyting space* is a triple $(X, \leq, R)$, where $(X, \leq)$ is a Heyting space and R is a binary relation in X that satisfies the following conditions:

(RF1) For every $U \in \mathcal{CU}(X, \leq)$ holds that $\{x \in X : R(x) \subseteq U\} \in \mathcal{CU}(X, \leq)$, where $R(x) = \{y \in X : xRy\}$;

(RF2) $R \subseteq \leq$;

(RF3) $< \subseteq R$.

Here $<$ is the strict order associated to the order $\leq$.

Morphisms of Rf-Heyting spaces are functions $g : (X_1, \leq, R_1) \to (X_2, \leq, R_2)$, where $g : (X_1, \leq) \to (X_2, \leq)$ is a morphism of Heyting spaces such that for every $U \in \mathcal{CU}(X_2, \leq)$ and $x \in X_1$ holds the following condition:

$$(C) \quad R_1(x) \subseteq g^{-1}(U) \Leftrightarrow R_2(g(x)) \subseteq U$$

The category **fSH** consists of all Rf-Heyting spaces and morphisms of Rf-Heyting spaces.

If X is a poset, for every $U \subseteq X$ we write U^c to indicate the set $\{x \in X : x \notin U\}$. Let $(X, \leq)$ be a Heyting space. For every $U, V \subseteq X$ we define the following subsets of X: $\downarrow U = \{x \in X : x \leq u, \text{ for some } u \in U\}$ and $U \to V = [\downarrow (U \cap V^c)]^c$. Let R be a binary relation in X. For every $U \subseteq X$ we define the following subset of X:

$$\tau_R(U) = \{x \in X : R(x) \subseteq U\}$$

We consider the following conditions, for every $U, V \in \mathcal{CU}(X, \leq)$:

(Rf2) $U \subseteq \tau_R(U)$,

(Rf3) $\tau_R(U) \subseteq V \cup (V \to U)$.

An easy computation proves that condition $(RF2)$ is equivalent to condition $(Rf2)$, and that condition $(RF3)$ implies condition $(Rf3)$.

We consider the contravariant functor $\mathcal{CU} : \mathbf{HS} \to \mathbf{HA}$ resricted to **fHS**.

We start with some preliminary lemmas.

Lemma 3.1. *Let $(X, \leq, R)$ be a Rf-space. Then $(\mathcal{CU}(X, \leq), \subseteq, \tau_R)$ is a frontal Heyting algebra.*

Proof. The well definition of τ_R is consequence of $(RF1)$. Conditions $(Rf2)$ and $(Rf3)$ give us the equations $(f2)$ and $(f3)$ respectively. Finally $(f1)$ is consequence of the definition of τ_R. □

***Remark* 3.2.** Let $g : (X_1, \leq) \to (X_2, \leq)$ be a morphism of Heyting spaces and $(X_1, \leq, R_1)$, $(X_2, \leq, R_2)$ Rf-Heyting spaces.

Then g is a Rf-morphism if and only if for every $U \in \mathcal{CU}(X, \leq)$ we have that

$$\tau_{R_1}(\mathcal{CU}(g)(U)) = \mathcal{CU}(g)(\tau_{R_2}(U))$$

Indeed,

$$g \text{ satisfies the condition } (C) \Leftrightarrow$$

$$\{x \in X : R_1(x) \subseteq g^{-1}(U)\} = \{x \in X : R_2(g(x)) \subseteq U\} \Leftrightarrow$$

$$\{x \in X : R_1(x) \subseteq \mathcal{CU}(g)(U)\} = \mathcal{CU}(g)(\{y \in Y : R_2(y) \subseteq U\}) \Leftrightarrow$$

$$\tau_{R_1}(\mathcal{CU}(g)(U)) = \mathcal{CU}(g)(\tau_{R_2}(U)).$$

Lemma 3.3. *Let $g : (X_1, \leq, R_1) \to (X_2, \leq, R_2)$ be a morphism of Rf-Heyting spaces. Then $\mathcal{CU}(g) : (\mathcal{CU}(X_2, \leq), \tau_{R_2}) \to (\mathcal{CU}(X_1, \leq), \tau_{R_1})$ is a frontal Heyting morphism.*

Proof. It is a consequence of Remark 3.2. □

The previous two lemmas show that $\mathcal{CU}$ is a contravariant functor from **fHS** to **fHA**. We now consider the contravariant functor $\mathcal{PF} : \mathbf{HA} \to \mathbf{HS}$ resricted to **fHA**.

Let H be a Heyting algebra and $A \subseteq H$. We will write $F(A)$ for the filter generated by A and $I(A)$ for the ideal generated by A.

Lemma 3.4. *Let (H, τ) be a frontal Heyting algebra and $P \in \mathcal{PF}(H)$. Then:*

(a) *$\tau^{-1}(P)$ is a filter.*

(b) *$\tau(x) \notin P \Leftrightarrow$ there exists $Q \in \mathcal{PF}(H)$ such that $\tau^{-1}(P) \subseteq Q$ and $x \notin Q$.*

Proof. (a) Straightforward.
(b) We suppose that $\tau(x) \notin P$, that is $x \notin \tau^{-1}(P)$. Then by (a) and by the Prime Filter Theorem, there is $Q \in \mathcal{PF}(H)$ such that $\tau^{-1}(P) \subseteq Q$ and $x \notin Q$. Conversely, let $Q \in \mathcal{PF}(H)$ such that $\tau^{-1}(P) \subseteq Q$ and $x \notin Q$. Then, $x \notin \tau^{-1}(P)$, so $\tau(x) \notin P$. □

Let (H, τ) be a frontal Heyting algebra. We define in $\mathcal{PF}(H)$ the following binary relation:

$$(P, Q) \in R_\tau \Leftrightarrow \tau^{-1}(P) \subseteq Q$$

Lemma 3.5. *Let* (H, τ) *be a frontal Heyting algebra. Then for every* $x \in H$ *we have that*

$$\varphi_H(\tau(x)) = \{P \in \mathcal{PF}(H) : R_\tau(P) \subseteq \varphi_H(x)\}$$

Proof. We have that, $R_\tau(P) \subseteq \varphi_H(x)$ is equivalent to $(\tau^{-1}(P) \subseteq Q \Rightarrow Q \in \varphi_H(x))$, and by definition of φ_H, this is equivalent to $(\tau^{-1}(P) \subseteq Q \Rightarrow x \in Q)$. By Lemma 3.4 this last expression is equivalent to $\tau(x) \in P$. Hence we conclude that $R_\tau(P) \subseteq \varphi_H(x) \Leftrightarrow P \in \varphi_H(\tau(x))$. $\square$

Lemma 3.6. *Let* (H, τ) *be a frontal Heyting algebra. Then* $(\mathcal{PF}(H), \subseteq, R_\tau)$ *is a* Rf*-space.*

Proof. $(RF1)$ Let U be a clopen upset in $(\mathcal{PF}(H), \subseteq)$, so there is $a \in H$ such that $U = \varphi_H(a)$. By Lemma 3.5 condition $(RF1)$ holds.

$(RF2)$ Let $P \in U$ and $Q \in R_\tau(P)$. By this reason $a \in P$ and $\tau^{-1}(P) \subseteq Q$. By $(f2)$ we have that $\tau(a) \in P$, so $a \in Q$. Then $Q \in U$, so $R_\tau(P) \subseteq U$. By this reason $(Rf2)$ holds.

$(RF3)$ Assume that $(RF3)$ does not hold. Then there exist $P, Q \in \mathcal{PF}(H)$ such that $P \subset Q$ and $\tau^{-1}(P) \nsubseteq Q$. and hence we have that there exist $x, y \in H$ such that $\tau(x) \in P$, $x \notin Q$, $y \in P$ and $y \notin Q$. Aplying $(f3)$ we have that $y \to x \in Q$ and using that $y \in Q$ we conclude that $x \in Q$, a contradiction. $\square$

Lemma 3.7. *Let* $f : (H_1, \tau_1) \to (H_2, \tau_2)$ *be a frontal Heyting morphism. Then* $\mathcal{PF}(f) : ((\mathcal{PF}(H_2), \subseteq), R_{\tau_2}) \to ((\mathcal{PF}(H_1), \subseteq), R_{\tau_1})$ *is a morphism of* Rf*-Heyting spaces.*

Proof. Let $X_i = \mathcal{PF}(H_i)$ and $R_i = R_{\tau_i}$ for $i = 1, 2$. We know that $\varphi_{H_2} f \varphi_{H_1}^{-1} = \mathcal{CU}(\mathcal{PF}(f))$. Let $U \in \mathcal{CU}(\mathcal{PF}(H_1), \subseteq)$, so there is $x \in H_1$ such that $U = \varphi_{H_1}(x)$. By Lemma 3.5 we have that $\varphi_{H_1}(\tau_1(x)) = \{P \in \mathcal{PF}(H_1) : R_1(P) \subseteq U\}$. We have already shown that $\varphi_{H_1}(\tau_1(x)) = \tau_{R_1}(U)$. So by hypothesis we conclude that

$$\mathcal{CU}(\mathcal{PF}(f))(\tau_{R_1}(U)) = (\varphi_{H_2} f \varphi_{H_1}^{-1})(\varphi_{H_1} \tau_1(x)) = \varphi_{H_2} f \tau_1(x) = \varphi_{H_2} \tau_2 f(x). \tag{4}$$

Besides, by Lemma 3.5 we have that

$$\tau_{R_2}(\mathcal{CU}(\mathcal{PF}(f))(U)) = \tau_{R_2}(\varphi_{H_2} f \varphi_{H_1}^{-1})\varphi_{H_1}(x) = \tau_{R_2}\varphi_{H_2} f(x) =$$

$$\{P \in \mathcal{PF}(H_2) : R_{\tau_2}(P) \subseteq \varphi_{H_2} f(x)\} = \varphi_{H_2}(\tau_2 f(x)). \tag{5}$$

By (4) and (5) we conclude that

$$\mathcal{CU}(\mathcal{PF}(f))(\tau_{R_1}(U)) = \tau_{R_2}(\mathcal{CU}(\mathcal{PF}(f))(U)) \tag{6}$$

By (6) and Remark 3.2 (taking $g = \mathcal{PF}(f)$) we have that $\mathcal{CU}(g)$ is a morphism in **fHS**. □

Last two previous lemmas show that $\mathcal{PF}$ is a contravariant functor from **fHA** to **fHS**. We will now see that these categories are dual equivalence to each other.

Proposition 3.8. *Let $(X, \leq, R)$ be a Rf-Heyting space. Then there exists an isomorphism of Rf-Heyting spaces $G_X : (X, \leq, R) \to (\mathcal{PF}(\mathcal{CU}(X, \leq), R_{\tau_R})$, given by*

$$G_X(x) = \{U \in \mathcal{CU}(X, \leq) : x \in U\}.$$

Proof. We write R_R in place of R_{τ_R}. For every clopen upset V in $\mathcal{PF}(\mathcal{CU}(X, \leq))$ we have to prove that

$$R(x) \subseteq G_X^{-1}(V) \Leftrightarrow R_R(G_X(x)) \subseteq V$$

($\Rightarrow$) Let $P \in R_R(G_X(x))$, so

$$\tau_R^{-1}(G(x)) \subseteq P \tag{7}$$

Then we have that

$$(7) \Leftrightarrow [U \in \tau_R^{-1}(G(x)) \Rightarrow U \in P] \Leftrightarrow [\tau_R(U) \in G_X(x) \Rightarrow U \in P] \Leftrightarrow$$
$$[x \in \tau_R(U) \Rightarrow U \in P] \Leftrightarrow [R(x) \subseteq U \Rightarrow U \in P]$$

But by hypothesis $R(x) \subseteq G_X^{-1}(V)$, so $G_X^{-1}(V) \in P$. As $P \in \mathcal{PF}(\mathcal{CU}(X, \leq))$ there is $y \in X$ such that $G_X(y) = P$. As $G_X^{-1}(V) \in G_X(y)$ we have that $y \in G_X^{-1}(V)$, so $G_X(y) = P \in V$.

($\Leftarrow$) Let $y \in R(x)$. Our hypothesis is equivalent to the condition $G_X^{-1} R_R(G_X(x)) \subseteq G_X^{-1}(V)$. If we will prove that $y \in G_X^{-1} R_R(G_X(x))$ then we would have that $y \in G_X^{-1}(V)$, which is our aim. We observe that

$$y \in G_X^{-1} R_R(G_X(x)) \Leftrightarrow G_X(x) R_R G_X(y) \Leftrightarrow \tau_R^{-1}(G_X(x)) \subseteq G_X(y) \Leftrightarrow$$

$$[U \in \tau_R^{-1}(G_X(x)) \Rightarrow U \in G_X(y)] \Leftrightarrow [\tau_R(U) \in G_X(x) \Rightarrow U \in G_X(y)] \Leftrightarrow$$
$$[x \in \tau_R(U) \Rightarrow y \in U] \Leftrightarrow [R(x) \subseteq U \Rightarrow y \in U]$$

As $y \in R(x)$, by the previous observation we have that $y \in G_X^{-1} R_R(G_X(x))$. □

Proposition 3.9. *Let (H, τ) be a frontal Heyting algebra. Then there is a frontal Heyting isomorphism $\varphi_H : (H, \tau) \to (\mathcal{CU}(\mathcal{PF}(H), \subseteq), \tau_{R_\tau})$, given by*

$$\varphi_H(h) = \{P \in \mathcal{PF}(H) : h \in P\}.$$

Proof. It follows from Lemma 3.5 □

Since both **fHA** and **fHS** are subcategories of **HA** and **HS** and $\mathcal{CU}$ and $\mathcal{PF}$ are the restriction to the former categories of an adjoint pair between **HA** and $\mathbf{HS}^{op}$, the following Theorem follows from Propositions 3.8 and 3.9.

Theorem 3.10. *Functors $\mathcal{CU}$ and $\mathcal{PF}$ establish a dual equivalence between the categories **fHA** and **fHS**.*

4. Applications of the duality

In this section we give some applications of the duality developed in section 3.

Let (H, τ) be a frontal Heyting algebra. We say that $M \subseteq H$ is a subalgebra of (H, τ) if M is a Heyting subalgebra of H and for each $x \in M$, $\tau(x) \in H$; i.e., a subalgebra in the category **fHA**. We characterize the relations associated with the subalgebras of a Heyting frontal algebra (H, τ). This result is based on the duality given in [3] and [5]. We also give easy descriptions of the representation theories of Heyting algebras that admit each of the operators studied in section 2.

4.1 Subalgebras

Let L be a bounded distributive lattice and M a sublattice of L. We define the binary relation

$$R_M = \{(P, Q) \in \mathcal{PF}(L) \times \mathcal{PF}(L) : Q \cap M \subseteq P\}$$

For each binary relation R in $\mathcal{PF}(L)$ we define the subset of clopen upsets of $\mathcal{PF}(L)$

$$M_R = \{U \in \mathcal{CU}(\mathcal{PF}(L), \subseteq) : R^{-1}(U) \subseteq U\}.$$

It was shown in [5] that M_R is a bounded sublattice of $\mathcal{CU}(\mathcal{PF}(L), \subseteq)$ and that the relation R_M is reflexive, transitive and when $X = \mathcal{PF}(L)$ it verifies that

(l) If for $P, Q \in \mathcal{PF}(L)$ such that $(P, Q) \notin R_M$ there exists $U \in M_R$ such that $P \in U$ and $Q \notin U$ then $(G_X(P), G_X(Q)) \notin R_{M_R}$.

It was also shown that the correspondence $M \longmapsto R_M$ establishes an anti-isomorphism between the lattice of bounded sublattices of a bounded distributive lattice L and the lattice of binary relations defined in the Priestley space $\mathcal{PF}(L)$ whose are reflexive, transitive and satisfies the condition (l).

Let X be a set, R_1 and R_2 binary relations in X. We define the binary relation $R = R_1 \circ R_2$ in the following way:

$$(x, y) \in R \Leftrightarrow \textit{ there exists } z \in X \textit{ such that } (x, z) \in R_1 \textit{ and } (z, y) \in R_2.$$

For any Heyting algebra H we write H_L for underlying bounded distributive lattice. Let (H, τ) be a frontal Heyting algebra and R, R_{M^*} the binary relations given by

$$R_{\tau^*} = \{(I, J) : (I^c, J^c) \in \mathcal{PF}(H_L) \times \mathcal{PF}(H_L) \textit{ and } J \subseteq \tau^{-1}(I)\},$$

$$R_{M^*} = \{(I, J) : (I^c, J^c) \in \mathcal{PF}(H_L) \times \mathcal{PF}(H_L) \textit{ and } J \cap M \subseteq I\}$$

We observe that $\mathcal{PF}(H) = \mathcal{PF}(H_L)$.

Lemma 4.1. *Let (H, τ) a frontal Heyting algebra and M a subalgebra of H_L.*

The following conditions are equivalent:

(i) $R_M^{-1} \circ R_\tau \subseteq R_\tau \circ R_M^{-1}$

(ii) $R_{M^*} \circ R_{\tau^*} \subseteq R_{\tau^*} \circ R_{M^*}$

Proof. ($(i) \Rightarrow (ii)$) Let $(I, J) \in R_{M^*} \circ R_{\tau^*}$. Then there exists $K^c \in \mathcal{PF}(H)$ such that $K \cap M \subseteq I$ and $J \subseteq \tau^{-1}(K)$. Thus $I^c \cap M \subseteq K^c$ and $\tau^{-1}(K^c) \subseteq J^c$. Then $(I^c, K^c) \in R_M^{-1}$ and $(K^c, J^c) \in R_\tau$, so $(I^c, J^c) \in R_M^{-1} \circ$

R_τ. Then by hypothesis there exists $P \in \mathcal{PF}(H)$ such that $(I^c, P) \in R_\tau$ and $(P, J^c) \in R_M^{-1}$, so $\tau^{-1}(I^c) \subseteq P$ and $P \cap M \subseteq J^c$. Therefore we have that $P^c \subseteq \tau^{-1}(I)$ and $J \cap M \subseteq P$. Thus $(I, J) \in R_{\tau^*} \circ R_M^*$.

($(ii) \Rightarrow (i)$) Let $(P, Q) \in R_M^{-1} \circ R_\tau$, so there exists $Z \in \mathcal{PF}(H)$ such that $P \cap M \subseteq Z$ and $\tau^{-1}(Z) \subseteq Q$. Then $Z^c \cap M \subseteq P^c$ and $Q^c \subseteq \tau^{-1}(Z^c)$. Thus $(P^c, Q^c) \in R_{M^*} \circ R_{\tau^*}$, so by hypothesis there exists $K^c \in \mathcal{PF}(H)$ such that $K \subseteq \tau^{-1}(P^c)$ and $Q^c \cap M \subseteq K$. Then $\tau^{-1}(P) \subseteq K^c$ and $K^c \cap M \subseteq Q$, so $(P, Q) \in R_\tau \circ R_M^{-1}$. □

For any bounded distributive lattice L we write L^* for the lattice with the same underlying set, but inverse order. We have the following result for subalgebras of a frontal Heyting algebra,

Theorem 4.2. *Let (H, τ) be a frontal Heyting algebra and M a subalgebra of H.*
Then the following conditions are equivalent:

(a) *M is a subalgebra of (H, τ)*

(b) $R_M^{-1} \circ R_\tau \subseteq R_\tau \circ R_M^{-1}$

Proof. On one hand we have that M is a sublattice of H_L^*. Consider the unary operator $j : H_L^* \to H_L^*$ given by $j(x) = \tau(x)$. Since τ is a frontal operator, j preserves $\vee$ and top in H_L^*. Hence M is a subalgebra of (H_L^*, j). Then, by ([3], Theorem 13) and Lemma 4.1 we conclude (b).

On the other hand, since M is a subalgebra of H, it is a sublattice of H_L. Consider again the operator j. By Lemma 4.1 and ([3], Theorem 13) we conclude that M is a subalgebra of (H_L^*, j), and then it is a subalgebra of (H, τ). □

Let H be a Heyting algebra. We consider the following binary relation in $\mathcal{PF}(H)$:

$$(P, Q) \in R_H \text{ iff for all } x,\, y \in H, \text{ if } x \to y \in P \text{ and } x \in Q, \text{ then } y \in Q.$$

The relation R_H is the relation of inclusion (Theorem 4.24 of [4]).

Corolary 4.3. *Let (H, τ) be a frontal Heyting algebra. The correspondence $M \longmapsto R_M$ establishes an anti-isomorphism from the lattice of subalgebras of (H, τ) and the lattice of binary relations defined in the Heyting space $\mathcal{PF}(H)$ whose are reflexive, transitive, satisfies the condition (l) and such that*

(1) $R_M^{-1} \circ R_H \subseteq R_H \circ (R_M^{-1} \cap R_M)$,

(2) $R_M^{-1} \circ R_\tau \subseteq R_\tau \circ R_M^{-1}$.

Proof. Consequence of ([4], Corollary 7.2) and Theorem 4.2. □

4.2 Representation theory for S-Heyting algebras

Let $(X, \leq)$ be a Heyting space and R a binary relation in X. We define the following condition for every $U \in \mathcal{CU}(X, \leq)$:

(RF4) If $x \notin U$ then there exists $y \in U^c$ such that $x \leq y$ and $R(y) \subseteq U$.

Condition $(RF4)$ is equivalent to the condition $\tau_R(U) \to U \subseteq U$ for every $U \in \mathcal{CU}(X, \leq)$.

In this case we write S_R in place of τ_R.

The category **SHS** is that whose objects are Rf-Heyting spaces $(X, \leq, R)$ that for every $U \in \mathcal{CU}(X, \leq)$ satisfy the condition $(RF4)$. Morphisms are the same of the category **fHS**.

Theorem 4.4. *There is a dual categorical equivalence between* ***SHA*** *and* ***SHS****.*

Proof. Consequence of Proposition 2.3 and Theorem 3.10. □

In what follows we will prove some results which will allow us to give an easy description of the category **SHS**.

We will say that a Heyting space $(X, \leq)$ is a S-Heyting space if for every $U \in \mathcal{CU}(X, \leq)$ the set $U \cup (U^c)_M$ is clopen, where $(U^c)_M$ is the set of maximal elements in U^c. We observe that $(X, \leq)$ is a S-Heyting space if and only if is a Heyting space such that for every clopen downset V the set V_M is clopen.

Lemma 4.5. *Let H be a Heyting algebra and V a clopen downset in $(\mathcal{PF}(H), \subseteq)$. Then $V = \downarrow (V_M)$.*

Proof. Let V be a downset. Then $\downarrow(V_M) \subseteq V$. Conversely, let $P \in V$. We have that $V = \varphi_H(x)^c$, for some $x \in H$. We consider the set

$$\Sigma = \{F \in \mathcal{PF}(H) : P \subseteq F,\ x \notin F\}.$$

By Zorn's Lemma there exists an element Q maximal in Σ. This Q is also maximal in V. $\square$

Corolary 4.6. *Let $(X, \leq)$ be a Heyting space and V a clopen downset in $(X, \leq)$. Then $V = \downarrow(V_M)$.*

Lemma 4.7. *Let $(X, \leq)$ be a Heyting space. If there exists a binary relation R in X that satisfies the conditions $(RF2)$, $(RF3)$ and $(RF4)$ then for every $U \in \mathcal{CU}(X, \leq)$ it holds that $S_R(U) = U \cup (U^c)_M$.*

Proof. Let $x \in S_R(U)$, $x \in U^c$ and $x \leq y$, with $y \in U^c$. Suppose that $y \nleq x$. Then there exists $V \in \mathcal{CU}(X, \leq)$ such that $y \in V$ and $x \notin V$. By $(Rf3)$ $x \in V \to U$. However, as $x \leq y$ with $y \in U^c \cap V$, we conclude that $x \notin V \to U$, a contradiction. For this reason $x \in (U^c)_M$.

Conversely, let $x \in U \cup (U^c)_M$. If $x \in U$, by $(Rf2)$ we have that $x \in S_R(U)$. If $x \in (U^c)_M$, by $(RF4)$ $x \notin S_R(U) \to U$. Therefore $x \leq y$ for some $y \in U^c$ and $R(y) \subseteq U$. But as $x \in (U^c)_M$ results that $x = y$. So $R(x) \subseteq U$. We have proved the equality $S_R(U) = U \cup (U^c)_M$. $\square$

Proposition 4.8. *Let $(X, \leq)$ be a Heyting space. There exists a binary relation R in X that satisfies the conditions $(RF1)$, $(RF2)$, $(RF3)$ and $(RF4)$ if and only if $(X, \leq)$ is a S-Heyting space.*

Proof. $(\Rightarrow)$ By $(RF1)$ and Lemma 4.7 we conclude that $U \cup (U^c)_M$ is clopen.
$(\Leftarrow)$ We define the following binary relation R in X:

$$xRy \Leftrightarrow (\forall V \in \mathcal{CU}(X, \leq))[x \in V \cup (V^c)_M \Rightarrow y \in V]$$

We will prove that R satisfies $(RF1)$, $(RF2)$, $(RF3)$ and $(RF4)$:

Let $U \in \mathcal{CU}(X, \leq)$, $x \in U$ and $y \in R(x)$. By definition of the relation R we have that $y \in U$ and for this reason $R(x) \subseteq U$. So we get that $(Rf2)$ holds.

Suppose that there exists $x, y \in X$ such that $x < y$ and that $y \notin R(x)$. Then there exists $U \in \mathcal{CU}(X, \leq)$ such that $x \in U \cup (U^c)_M$ and $y \notin U$. Thus

$x \notin U$, so $x \in (U^c)_M$. However $x < y$ and $y \notin U$, a contradiction with the maximality of x. Then $(RF3)$ holds.

Let $U \in \mathcal{CU}(X, \leq)$ and $x \in U^c$. By Corolary 4.6 there exists $y \in (U^c)_M$ such that $x \leq y$. In particular $x \leq y$ and $y \in U^c$. Besides $R(y) \subseteq U$. Let $z \in R(y)$. Since $y \in (U^c)_M$ it holds that $z \in U$. For this reason $x \notin S_R(U) \to U$, and $(RF4)$ follows.

By Lemma 4.7 we conclude that for every $U \in \mathcal{CU}(X, \leq)$ we have that $S_R(U) = U \cup (U^c)_M$. By hypothesis results that $S_R(U)$ is clopen and by definition this set is an upset. So we have $(RF1)$. □

It follows from the proof of previous lemma that if $(X, \leq, R)$ is an object of **SHS** then for every $U \in \mathcal{CU}(X, \leq)$, $S(U) = S_R(U) = U \cup (U^c)_M$ and also that if $(X, \leq)$ is a S-Heyting space then S exists in $\mathcal{CU}(X, \leq)$ and it is given by the formula

$$S(U) = U \cup (U^c)_M.$$

Let $(X, \leq)$ and $(Y, \leq)$ be S-Heyting spaces and $g : (X, \leq) \to (Y, \leq)$ a Heyting morphism. We will say that g is a S-Heyting morphism if for every V downset in $(Y, \leq)$

$$g^{-1}(V_M) = [g^{-1}(V)]_M$$

Proposition 4.9. *Let $g : (X, \leq) \to (Y, \leq)$ be a Heyting morphism of Heyting spaces. There are binary relations R_1 and R_2 in X and Y respectively such that the function $g : (X, \leq, R_1) \to (Y, \leq, R_2)$ is a morphism in **SHS** if and only if g is a S-Heyting morphism.*

Proof. It is a consequence of Remark 3.2, Proposition 4.8 and the following fact: $g^{-1}[U \cup (U^c)_M] = g^{-1}(U) \cup [g^{-1}(U^c)]_M$ if and only if $[g^{-1}(U^c)]_M = g^{-1}[(U^c)_M]$. □

Let $\mathbf{SH}_S$ be the category whose objects are S-Heyting spaces and whose morphisms are S-Heyting morphisms.

Theorem 4.10. *There exists an isomorphism of categories between **SHS** and $\mathbf{SH}_S$.*

Proof. It is a consequence of Propositions 4.8 and 4.9. □

4.3 Representation theory for γ-Heyting algebras

Let $(X, \leq)$ be a Heyting space and R a binary relation in X. We define the following conditions:

($R\gamma_4$) For every $x \in X$ there exists $y \in X$ such that $x \leq y$ and $R(y) = \emptyset$.

($R\gamma_5$) For every $U \in \mathcal{CU}(X, \leq)$, if $R(x) \subseteq U$ then $R(x) = \emptyset$ or $x \in U$.

Conditions $(R\gamma_4)$ and $(R\gamma_5)$ are respectively equivalent to the following ones:

(i) $\neg\tau_R(\emptyset) = \emptyset$, for every $x \in X$;

(ii) $\tau_R(U) \subseteq U \cup \tau_R(\emptyset)$, for every $U \in \mathcal{CU}(X, \leq)$.

In this case we write γ_R in place of τ_R.

The category γ**SH** is that whose objects are Rf-Heyting spaces $(X, \leq, R)$ that satisfy the conditions $(R\gamma_4)$ and $(R\gamma_5)$. Morphisms are the same of the category **fHS**.

Theorem 4.11. *There is a dual categorical equivalence between γ**HA** and γ**HS**.*

Proof. It is a consequence of Proposition 2.4 and Theorem 3.10. □

In the following we will prove some results which will allow us to give an easy description of the category γ**HA**.

We will say that a Heyting space $(X, \leq)$ is a γ-Heyting space if $(X, \leq)$ is a Heyting space and for every $U \in \mathcal{CU}(X, \leq)$ the set $U \cup X_M$ is clopen. We observe that $(X, \leq)$ is a γ-Heyting space if and only if X_M is clopen.

Lemma 4.12. *Let $(X, \leq)$ be a Heyting space. If there exists a binary relation R in X that satisfies the conditions $(RF2)$, $(RF3)$, $(R\gamma_4)$ and $(R\gamma_5)$ then for every $U \in \mathcal{CU}(X, \leq)$ we have that*

$$\gamma_R(U) = U \cup X_M.$$

Proof. Let $A = \{x \in X : R(x) = \emptyset\}$. Let $x \in A$ and $y \in X$ such that $x \leq y$. We suppose that $y \not\leq x$. Then there exists $V \in \mathcal{CU}(X, \leq)$ such that $y \in V$ and $x \in V^c$. So by $(Rf3)$ we have that $x \in (\downarrow V)^c$. On the other hand $x \leq y$, where $y \in V$, and then $x \in \downarrow V$, a contradiction. We conclude that $x = y$ and therefore $x \in X_M$.

Conversely, let $x \in X_M$. So by $(R\gamma_4)$ there exists $y \in X$ such that $R(y) = \emptyset$ and $x \leq y$, but as $x \in X_M$ results that $x = y$, and then $R(x) = \emptyset$. We have proved that $A = X_M$. By $(Rf2)$ and $(R\gamma_5)$, for every $U \in \mathcal{CU}(X, \leq)$ we have that $\gamma_R(U) = U \cup X_M$. □

Proposition 4.13. *Let $(X, \leq)$ be a Heyting space. There exists a binary relation R in X such that satisfies the conditions $(RF1)$, $(RF2)$, $(RF3)$, $(R\gamma_4)$ and $(R\gamma_5)$ if and only if $(X, \leq)$ is a γ-Heyting space.*

Proof. $(\Rightarrow)$ By $(RF1)$ and Lemma 4.12 we conclude that for every $U \in \mathcal{CU}(X, \leq)$ the set $U \cup X_M$ is clopen.
$(\Leftarrow)$ We define R in the following way:

$$xRy \Leftrightarrow (\forall V \in \mathcal{CU}(X, \leq))[x \in V \cup X_M \Rightarrow y \in V]$$

We will prove that R satisfies $(RF1)$, $(RF2)$, $(RF3)$, $(R_{\gamma 4})$ and $(R_{\gamma 5})$.

Same ideas as in the proof of Proposition 4.8 prove $(RF2)$ and $(RF3)$.

In order to prove $(R\gamma_4)$, take $x \in X$. By Corolary 4.6, there exists $y \in X_M$ such that $x \leq y$. Let us see that $R(y) = \emptyset$. Take $z \in R(y)$, then $y \in (X_M \cup \emptyset)$, and hence, $z \in \emptyset$ which is a contradiction.

To prove $(R\gamma_5)$, suppose that $R(x) \subseteq U$. If $x \notin U$ then $x \notin R(x)$ and so $x \in X_M$. We can then conclude that $R(x) = \emptyset$.

Finally, by Lemma 4.12 we have that for every $U \in \mathcal{CU}(X, \leq)$, $\gamma_R(U) = U \cup X_M$. Then $\gamma_R(U)$ is clopen, and by definition an upset. Hence we have $(RF1)$. □

As a consequence of previous proof, we have that if $(X, \leq, R)$ is an object of $\gamma\mathbf{HS}$ then for every $U \in \mathcal{CU}(X, \leq)$, $\gamma(U) = \gamma_R(U) = U \cup X_M$, and that if $(X, \leq)$ is a γ-Heyting space, γ exists in $\mathcal{CU}(X, \leq)$ and it is given by the formula

$$\gamma(U) = U \cup X_M.$$

Let $(X, \leq)$ and $(Y, \leq)$ be γ-Heyting spaces and $g : (X, \leq) \to (Y, \leq)$ a Heyting morphism. We will say that g is a γ-Heyting morphism

$$X_M = g^{-1}(Y_M)$$

Proposition 4.14. *Let $g : (X, \leq) \to (Y, \leq)$ a Heyting morphism of Heyting spaces.*

*There are binary relations R_1 y R_2 in X and Y respectively such that the function $g : (X, \leq, R_1) \to (Y, \leq, R_2)$ is a morphism in γ**HS** if and only if g is a γ-Heyting morphism.*

Proof. It is a consequence of Remark 3.2, Proposition 4.13 and the following fact:
$g^{-1}[U \cup Y_M] = g^{-1}(U) \cup X_M$ if and only if $X_M = g^{-1}(Y_M)$. □

Let $\mathbf{SH}_\gamma$ be the category whose objects are γ-Heyting spaces and whose morphisms are γ-Heyting morphisms.

Theorem 4.15. *There exists an isomorphism of categories between γ**HS** and $\boldsymbol{SH}_\gamma$.*

Proof. It is a consequence of Propositions 4.13 and 4.14. □

4.4 Representation theory for G-Heyting algebras

Let $(X, \leq)$ be a Heyting space and R a binary relation in X. We define the following conditions:

(RG4) $(\forall U \in \mathcal{CU}(X, \leq))[R(x) \subseteq U \Rightarrow \forall y \geq x \ \exists u \in U : y \leq u]$.

(RG5) If $x \leq y$, $y \in U^c$ and $(\forall y \geq x \ \exists u \in U : y \leq u)$ then there exists $z \in U^c$ such that $R(z) \subseteq U$ and $x \leq z$.

Conditions $(RG4)$ and $(RG5)$ are respectively equivalent to the following ones:

(i) $\tau_R(U) \subseteq \neg\neg U$, for every $U \in \mathcal{CU}(X, \leq)$.

(ii) $\tau_R(U) \to U \subseteq \neg\neg U \to U$, for every $U \in \mathcal{CU}(X, \leq)$.

In this case we write G_R in place of τ_R.

The category **GHS** is that whose objects are Rf-Heyting spaces $(X, \leq, R)$ that satisfy conditions $(RG4)$ and $(RG5)$. Morphisms are the obvious ones.

Theorem 4.16. *There is a dual categorical equivalence between **GHA** and **GHS**.*

Proof. It is a consequence of Remark 2.6 and Theorem 3.10. $\square$

We now give some results that allow us to give an easy description of the category **GHS**.

We say that $(X, \leq)$ is a G-Heyting space if it is a Heyting space and for every $U \in \mathcal{CU}(X, \leq)$, the set $U \cup [\neg\neg U \cap (U^c)_M]$ is clopen. In an equivalent way, $(X, \leq)$ is a G-Heyting space if it is a Heyting space such that for every $U \in \mathcal{CU}(X, \leq)$ the set $\neg\neg U \cap (U^c)_M$ is clopen.

Lemma 4.17. *Let H be a Heyting algebra and $(X, \leq) = (\mathcal{PF}(H), \subseteq)$. Define in $(X, \leq)$ the following binary relation:*

$$PRQ \Leftrightarrow (\forall U \in \mathcal{CU}(X, \leq))[P \in U \cup [\neg\neg U \cap (U^c)_M] \Rightarrow Q \in U]$$

For every $P \in \mathcal{PF}(H)$ we have that $R(P) \neq \emptyset$.

Proof. For $P \in \mathcal{PF}(H)$ we define the filter $M = \{y \in H : \neg\neg y \in P\}$ and then the filter $F = F(P \cup M)$. We have that $0 \notin F$. Suppose that $0 \in F$. Then, there are $p \in P$ and $m \in M$ such that $p \leq \neg m$, and hence $\neg m \in P$. Since $m \in M$, we have that $\neg\neg m \in P$ and so $0 \in P$, a contradiction, because P is prime. Then by the Prime Filter Theorem there exists $Q \in \mathcal{PF}(H)$ such that $P \subseteq F \subseteq Q$. Let $U \in \mathcal{CU}(X, \leq)$ be such that $P \in U \cup [\neg\neg U \cap (U^c)_M]$. In particular there exists $x \in H$ such that $\varphi_H(x) = U$. Thus $Q \in U$. We have then proved that $R(P) \neq \emptyset$. $\square$

Corolary 4.18. *Let $(X, \leq)$ be a Heyting space and R the following binary relation defined on X:*

$$xRy \Leftrightarrow (\forall U \in \mathcal{CU}(X, \leq))[x \in U \cup [\neg\neg U \cap (U^c)_M] \Rightarrow y \in U]$$

Then for every $x \in X$ we have that $R(x) \neq \emptyset$.

Lemma 4.19. *Let $(X, \leq)$ be a Heyting space. If there exists a binary relation R in X that satisfies $(RF2)$, $(RF3)$, $(RG4)$ and $(RG5)$ then for every $U \in \mathcal{CU}(X, \leq)$ we have that*

$$G_R(U) = U \cup [\neg\neg U \cap (U^c)_M].$$

Proof. Let $U \in \mathcal{CU}(X, \leq)$, $R(x) \subseteq U$, $x \in U^c$ and $x \leq y$, with $y \in U^c$. Suppose that $y \nleq x$. Then there exists $V \in \mathcal{CU}(X, \leq)$ such that $y \in V$ and

$x \notin V$. By $(Rf3)$, $x \in V \to U$. However, since $x \leq y$ with $y \in U^c \cap V$, we conclude that $x \notin V \to U$, a contradiction. Hence $x \in (U^c)_M$. On the the other hand, from $(RG4)$ we conclude that $x \in \neg\neg U$.

Conversely, take $x \in U \cup [\neg\neg U \cap (U^c)_M]$. If $x \in U$, by $(Rf2)$, we have that $x \in G_R(U)$. If $x \in (U^c)_M \cap \neg\neg U$ we have by $(RG5)$ that there exists $y \in U^c$ such that $x \leq y$ and $R(y) \subseteq U$. Hence $x = y$ and $R(x) \subseteq U$. $\square$

Lemma 4.20. *Let $(X, \leq)$ be a Heyting space. There exists a binary relation R in X that satisfies $(RF1)$, $(RF2)$, $(RF3)$, $(RG4)$ and $(RG5)$ if and only if $(X, \leq)$ is a G-Heyting space.*

Proof. $(\Rightarrow)$ By $(RF1)$ and Lemma 4.19 we conclude that for every $U \in \mathcal{CU}(X, \leq)$ the set $U \cup [\neg\neg U \cap (U^c)_M]$ is clopen.

$(\Leftarrow)$ Define the following binary relation R in X:

$$xRy \Leftrightarrow (\forall V \in CU(X, \leq))[x \in V \cup [\neg\neg V \cap (V^c)_M] \Rightarrow y \in V]$$

We prove that R satisfies $(RF1)$, $(RF2)$, $(RF3)$, $(RF4)$ and $(RG5)$.

Same ideas as in the proof of Proposition 4.8 prove $(RF2)$ and $(RF3)$.

In order to prove $(RG4)$, take $U \in \mathcal{CU}(X, \leq)$ and $R(x) \subseteq U$. Suppose that $x \notin \neg\neg U$. Then $x \in \downarrow (\neg U)$. Hence there exists $y \in \neg U$ such that $x \leq y$. In particular, $y \notin U$. By hypothesis, $y \notin R(x)$. Then, there exists $V \in \mathcal{CU}(X, \leq)$ such that $x \in V \cup [(\neg\neg V \cap (V^c)_M]$ and $y \notin V$. Since $x \leq y$, we conclude that $x = y$. Since $y \in \neg U$, $x \in \neg U$.

On the other hand, by Corolary 4.18 there exists $z \in R(x)$. Hence, by hypothesis, $z \in U$. Since $x \in \neg U$ and $z \in R(x)$, $z \in \neg U$. Then $U \cap \neg U \neq \emptyset$, a contradiction.

To prove $(RG5)$ suppose that $U \in \mathcal{CU}(X, \leq)$ and $x \notin \neg\neg U \to U$. Then $x \leq z$ for some $z \in \neg\neg U \cap U^c$. By Corolary 4.6 there exists $y \in (U^c)_M$ such that $x \leq z \leq y$. Let us see that $R(y) \subseteq U$. Take $w \in R(y)$. Since $z \leq y$, $y \in \neg\neg U \cap (U^c)_M$. We conclude that $w \in U$.

Finally, by Lemma 4.19 we have that for every $U \in \mathcal{CU}(X, \leq)$, $G_R(U) = U \cup [\neg\neg U \cap (U^c)_M]$. Using the hypothesis we conclude that $G_R(U)$ is clopen. It can be shown that $G_R(U)$ is an upset. Thus, we have $(RF1)$. $\square$

As a consequence of the proof of previous Lemma, we have that if $(X, \leq, R)$ is an object of **GHS** then for every $U \in \mathcal{CU}(X, \leq)$, $G(U) = G_R(U) = U \cup [\neg\neg U \cap (U^c)_M]$ and that if $(X, \leq)$ is a G-Heyting space we have that G exists in $\mathcal{CU}(X, \leq)$ and

$$G(U) = U \cup [\neg\neg U \cap (U^c)_M]$$

Let $(X, \leq)$ and $(Y, \leq)$ be G-Heyting spaces and $g : (X, \leq) \to (Y, \leq)$ a Heyting morphism. We will say that g is a G-Heyting morphism if for every V downset in $(Y, \leq)$

$$g^{-1}[\neg\neg(V^c) \cap V_M] = g^{-1}[\neg\neg(V^c)] \cap [g^{-1}(V)]_M.$$

Proposition 4.21. *Let $g : (X, \leq) \to (Y, \leq)$ Heyting morphism of Heyting spaces. There are binary ralations R_1 and R_2 in X and Y respectively such that the function $g : (X, \leq, R_1) \to (Y, \leq, R_2)$ is a morphism in **GHS** if and only if g is a G-Heyting morphism.*

Proof. It is a consequence of Remark 3.2, Proposition 4.20 and the following fact:

$$g^{-1}[U \cup (\neg\neg U \cap (U^c)_M)] = g^{-1}(U) \cup [g^{-1}(\neg\neg U) \cap [(g^{-1}(U^c)]_M]$$

if and only if

$$[g^{-1}(U^c)]_M \cap g^{-1}(\neg\neg U) = g^{-1}[(U^c)_M \cap \neg\neg U].$$

□

Let $\mathbf{SH}_G$ be the category whose objects are G-Heyting spaces and whose morphisms are G-Heyting morphisms.

Theorem 4.22. *There exists an isomorphism of categories between **GHS** and $\boldsymbol{SH}_G$.*

Proof. It is a consequence of Propositions 4.20 and 4.21. □

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Departamento de Matemática,
Facultad de Ciencias Exactas, UNLP.
Casilla de correos 172,
La Plata (1900)
Argentina.

jlc@mate.unlp.edu.ar,
marta@mate.unlp.edu.ar,
hsanmartin@mate.unlp.edu.ar