Domagoj VRGOČ, Mladen VUKOVIĆ

BISIMULATION QUOTIENTS
OF VELTMAN MODELS

A b s t r a c t. Interpretability logic is a modal description of the interpretability predicate. The modal system IL is an extension of the provability logic GL (Gödel–Löb). Bisimulation quotients and largest bisimulations have been well studied for Kripke models. We examine interpretability logic and consider how these results extend to Veltman models.

1. Introduction

The idea of treating a provability predicate as a modal operator goes back to Gödel. The same idea was taken up later by Kripke and Montague, but only in the mid–seventies was the correct choice of axioms, based on Löb's theorem, seriously considered by several logicians independently: G. Boolos, D. de Jongh, R. Magari, G. Sambin and R. Solovay.

The system GL (Gödel, Löb) is a modal propositional logic. The axioms of system GL are all tautologies, $\Box(A \to B) \to (\Box A \to \Box B)$, and

$\Box(\Box A \rightarrow A) \rightarrow \Box A$. The inference rules of GL are modus ponens and necessitation $A/\Box A$. R. Solovay in 1976. proved arithmetical completeness of modal system GL. Many theories have the same provability logic - GL. Provability logic of Peano arithmetic, Zermelo–Fraenkel set theory and Gödel-Bernays set theory is the system GL. It means that provability logic GL cannot distinguish some properties, as e.g. finite axiomatizability, reflexivity, interpretabiltiy principles etc.

Roughly, a theory S interprets a theory T if there is a natural way of translating the language of T into the language of S in such a way that the translations of all the axioms of T become provable in S. We write $S \geq T$ if this is the case. A derived notion is that of relative interpretability over a base theory T. Let A and B be arithmetical sentences. We say that A interprets B over T if $T + A \geq T + B$. For essentially reflexive theories, such as Peano arithmetic and its extensions in the same language, the notion of relative interpretability coincides with that of Π_1-conservativity. For precise definitions and details, see e.g. [8].

Modal logics for interpretability were first studied by P. Hájek (1981) and V. Švejdar (1983). A. Visser (1990; see [7]) introduced the binary modal logic IL (interpretability logic). The interpretability logic IL results from the provability logic GL, by adding the binary modal operator $\rhd$.

The language of the interpretability logic contains propositional letters p_0, $p_1, \ldots$, logical connectives $\wedge$, $\vee, \rightarrow$ and $\neg$, unary modal operator $\Box$ and binary modal operator $\rhd$. We use $\bot$ for false and $\top$ for true. The axioms of the interpretability logic IL are all axioms of the system GL and $\Box(A \rightarrow B) \rightarrow (A \rhd B)$, $(A \rhd B \wedge B \rhd C) \rightarrow (A \rhd C)$, $((A \rhd C) \wedge (B \rhd C)) \rightarrow ((A \vee B) \rhd C)$, $(A \rhd B) \rightarrow (\Diamond A \rightarrow \Diamond B)$, and $\Diamond A \rhd A$, where $\Diamond$ stands for $\neg\Box\neg$ and $\rhd$ has the same priority as $\rightarrow$. The deduction rules of IL are modus ponens and necessitation.

Arithmetical semantic of interpretability logic is based on the fact that each sufficiently strong theory S has arithmetical formulas $Pr(x)$ and $Int(x, y)$; formula $Pr(x)$ expressing that "x is provable in S" (i.e. formula with Gödel number x is provable in S) and formula $Int(x, y)$ expressing that "$S + x$ interprets $S + y$." An arithmetical interpretation is a function $*$ from modal formulas into arithmetical sentences preserving Boolean connectives and satisfying $(\Box A)^* = Pr(\ulcorner A^* \urcorner)$, and $(A \rhd B)^* = Int(\ulcorner A^* \urcorner, \ulcorner B^* \urcorner)$. ($\ulcorner A^* \urcorner$ denotes Gödel number of formula A^*). A modal formula A is valid in a theory S if $S \vdash A^*$ for each arithmetical interpretation $*$. A modal theory

T is sound w.r.t. S if all its theorems are valid in S. The theory T is complete w.r.t. S if it proves exactly those formulas that are valid in S. The soundness of IL was already known and amounts to noticing that all the axioms are PA–valid and the rules of inference preserve PA-validity.

The system IL is natural from the modal point of view, but arithmetically incomplete. For example, IL does not prove the formula W i.e. $(A \rhd B) \to (A \rhd (B \wedge \Box(\neg A)))$, which is valid in every adequate theory. Various extensions of IL are obtained by adding some new axioms. These new axioms are called the principles of interpretability.

We are only interested in IL as a system of modal logic. There are several kinds of semantics for the interpretability logic. The basic semantic is Veltman models. De Jongh and Veltman proved in [4] the completeness of IL w.r.t. Veltman models. We think that there are two main reasons for other semantics. First, the proofs of arithmetical completeness of interpretability logic are very complex. Second, the characteristic classes Veltman frames of some principles of interpretability are equal. A. Berarducci proved the arithmetical completeness of system ILM (see [1]). System ILM is the interpretability logic of Peano arithmetic. Berarducci did not use Veltman models only. If the system ILM does not prove a formula A then de Jongh-Veltman's theorem gives the existence of a Veltman model W satisfying $W \not\models A$. By means of a bisimulation we can construct a Visser model (or a simplified Veltman model) W' such that the formula A is not true in model W'. Finally, by using Visser model W' Berarducci defined an arithmetical interpretation $*$ such that the formula A^* is not provable in Peano arithmetic. Visser (in [7]) proved the arithmetical completeness of the system ILP. Visser did not use Veltman models only. He used Friedman models, too. Generalized Veltman models were defined by de Jongh. R. Verbrugge, M. Vuković, E. Goris and J. Joosten used generalized Veltman models for proofs of independences of interpretability principles.

We consider here Veltman models and bisimulations. Let K and K' be two Kripke models. If we want to study correspondence between these models we consider an isomorphism or an elementarily equivalence. If we want to study "weaker" (but very useful) correspondence we can consider a bisimulation. Roughly speaking, a bisimulation is a subset Z of $K \times K'$. The basic property of each bisimulation Z is:

$$\text{if } \ vZw \ \text{ then } \ v \Vdash A \ \text{ if and only if } \ w \Vdash A,$$

for every formula A.

The notion of bisimulation is very important. A. Visser in [7] defined bisimulation of Veltman models and proved that every Veltman model satisfying: $xRyRzS_xu \Rightarrow zS_yu$, can be bisimulated by a finite Friedman model. This fact and de Jongh–Veltman's theorem imply completeness of the system ILP w.r.t. finite Friedman models. In the paper [1] Berarducci used a bisimulation for the proof of completeness of system IL w.r.t. simplified Veltman models. By using a bisimulation Visser (see [8]) proved that Craig interpolation lemma is not true for systems between ILM_0 and ILM. R. de Jonge proved in [3] Hennessy–Milner theorem for Veltman semantics. He defined the quotient structure $\overline{W}$ of a Veltman model W with respect to reflexive, symmetric and transitive autobisimulation. R. de Jonge constructed a Veltman model W such that the quotient mapping $\pi : W \to \overline{W}$ is not a bisimulation between models W and $\overline{W}$. For this reason, he introduced the notion of strong bisimulation. It was proved in [3] that every strong bisimulation is a bisimulation and that the quotient mapping from W to $\overline{W}$ is a bisimulation, provided that the starting autobisimulation Z is a strong, reflexive, transitive and symmetric bisimulation.

Let W_1 and W_2 be two Veltman models, and Z_1 and Z_2 strong autobisimulations on those models that are also equivalence relations. We prove here that the quotient structures $\overline{W_1}$ and $\overline{W_2}$ (with respect to Z_1 and Z_2) are isomorphic if and only if the models W_1 and W_2 are globally bisimilar.

The largest bisimulation between Kripke models are considered in [2] and [5]. At the end of this article we give some remarks on the existence of the largest bisimulation of Veltman model.

2. Bismulation quotient of Veltman models

The notion of Veltman model is defined in [4].

Definition 1. An ordered triple $\langle W, R, \{S_w : w \in W\}\rangle$ is called a Veltman frame if it satisfies the following conditions:

a) $\langle W, R\rangle$ is a GL-frame, i.e. W is a non-empty set, and R is transitive and reverse well-founded relation on W;

b) For every $w \in W$ is $S_w \subseteq W[w] \times W[w]$, where $W[w] = \{u : wRu\}$;

c) The relation S_w is reflexive and transitive for every $w \in W$;

d) If $wRuRv$ then uS_wv.

An ordered quadruple $\langle W, R, \{S_w : w \in W\}, \Vdash \rangle$ is called a Veltman model if it satisfies the following conditions:

1) $\langle W, R, \{S_w : w \in W\} \rangle$ is a Veltman frame;

2) $\Vdash$ is a forcing relation. We emphasize only the definition

$w \Vdash A \rhd B$ if and only if $\forall u((wRu \ \& \ u \Vdash A) \Rightarrow \exists v(uS_wv \ \& \ v \Vdash B))$.

We denote a Veltman model $\langle W, R, \{S_w : w \in W\}, \Vdash \rangle$ shortly by W.

D. de Jongh and F. Veltman in [4] proved completeness of the system IL w.r.t Veltman semantics. A. Visser defined the notion of bisimulation between Veltman models in [7].

Definition 2. A bisimulation between two Veltman models W and W' is a nonempty binary relation $Z \subseteq W \times W'$ such that the following conditions hold:

  (at)   If wZw' then $W, w \Vdash p$ if and only if $W', w' \Vdash p$, for all propositional variables p;

(forth)   If wZw' and wRu, then there exists $u' \in W'$ with $w'R'u'$, uZu' and for all $v' \in W'$ if $u'S_{w'}v'$ there is $v \in W$ such that uS_wv;

 (back)   If wZw' and $w'Ru'$, then there exists $u \in W$ with wRu, uZu' and for all $v \in W$ if uS_wv there is $v' \in W'$ such that $u'S_{w'}v'$.

If $W = W'$ we will call Z an autobisimulation. If $w \in W$ and Z is an autobisimulation on W, with $\overline{w}$ we denote the set $\{u \in W : wZu\}$.

In [3] de Jonge defined the quotient structure of Veltman model W with respect to reflexive, symmetric and transitive autobisimulation Z on W as $\overline{W} = \langle \overline{W}, \overline{R}, \{S_{\overline{w}} : \overline{w} \in \overline{W}\}, \Vdash \rangle$, where

$$\overline{W} = \{\overline{w} : w \in W\};$$

$$\overline{w}\overline{R}\overline{u} \Longleftrightarrow (\exists x \in \overline{w})(\exists y \in \overline{u})xRy;$$

$$\overline{u}\overline{S}_{\overline{w}}\overline{v} \Longleftrightarrow (\exists x \in \overline{w})(\exists y \in \overline{u})(\exists z \in \overline{v})yS_xz;$$

$$\overline{w} \Vdash p \Longleftrightarrow w \Vdash p.$$

It is easy to see that all the relations are well defined. It is also easy to show that $\langle \overline{W}, \overline{R}\rangle$ is a GL-frame, provided that $\langle W, R\rangle$ is. However, we construct an example showing that the quotient structure as defined by de Jonge does not give a Veltman model. The only property that fails with this definition is the transitivity of S_w.

To see this consider the following Veltman model:
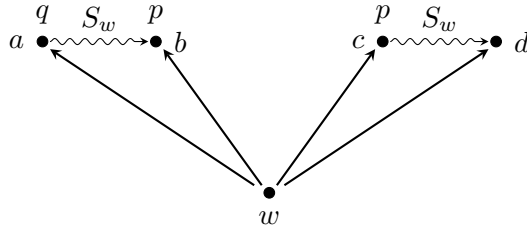
$$W = \{w, a, b, c, d\}$$

$$R = \{(w, a), (w, b), (w, c), (w, d)\}$$

$$S_w = \{(a, b), (c, d)\}$$

$$V(p) = \{b, c\} \ , \ V(q) = \{a\}.$$

It is easy to check that $Z = \{(x, x) : x \in W\} \cup \{(b, c), (c, b)\}$ is indeed an autobisimulation on W and that it is reflexive, transitive and symmetric.
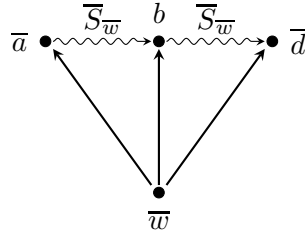
If we use the definition of quotient proposed by de Jonge, we get the following structure:

$$\overline{W} = \{\overline{w}, \overline{a}, \overline{b}, \overline{d}\}$$

$$\overline{R} = \{(\overline{w}, \overline{a}), (\overline{w}, \overline{b}), (\overline{w}, \overline{d})\}$$

$$\overline{S}_{\overline{w}} = \{(\overline{a}, \overline{b}), (\overline{b}, \overline{d})\}$$

$$V(p) = \{\overline{b}\} \ , \ V(q) = \{\overline{a}\}.$$

Now note that we have $\overline{a}\,\overline{S_{\overline{w}}}\,\overline{b}$ and $\overline{b}\,\overline{S_{\overline{w}}}\,\overline{d}$, but we don't have $\overline{a}\,\overline{S_{\overline{w}}}\,\overline{d}$ because $aS_w d$ is not satisfied. We conclude that $\overline{S_{\overline{w}}}$ is not transitive and in turn, the quotient structure $\overline{W}$ is not a Veltman model.

An easy way to correct this is to take the transitive closure of the relation defined by de Jonge. Formally, we will replace the definition of $\overline{S_{\overline{w}}}$ with the following:

$$\overline{u}\,\overline{S}'_{\overline{w}}\,\overline{v} \iff (\exists x \in \overline{w})(\exists y \in \overline{u})(\exists z \in \overline{v})yS_x z;$$

$$\overline{S_{\overline{w}}} = (\overline{S}'_{\overline{w}})^{+}.$$

Here $(\overline{S}'_{\overline{w}})^{+}$ denotes the transitive closure of $\overline{S}'_{\overline{w}}$.

It is easy to check that $\langle \overline{W}, \overline{R}, \{\overline{S_{\overline{w}}} : \overline{w} \in \overline{W}\}, \Vdash \rangle$ is indeed a Veltman model.

In [3] de Jonge constructed a model W such that the quotient mapping $\pi : W \to \overline{W}$ defined by $\pi(w) = \overline{w}$ is not a bisimulation between W and $\overline{W}$. In fact, he showed that the two models aren't even modally equivalent. Even if we consider the corrected definition of the quotient model (with $\overline{S_{\overline{w}}}$ now transitive) this still remains true.

For this reason, in [3], de Jonge introduced the notion of strong bisimulation.

Definition 3. A strong bisimulation between two Veltman models W and W' is a nonempty binary relation $Z \subseteq W \times W'$ such that the following conditions hold:

(at) If wZw' then $W, w \Vdash p$ if and only if $W', w' \Vdash p$, for all propositional variables p;

(forth1) If wZw' and wRu, then there exists $u' \in W'$ with $w'R'u'$ and uZu';

(forth2) If wZw', uZu', $w'R'u'$ and $uS_w v$ then there is $v' \in W'$ with vZv' and $u'S_{w'}v'$;

(back1) If wZw' and $w'Ru'$, then there exists $u \in W$ with wRu and uZu';

(back2) If wZw', uZu', wRu and $u'S_{w'}v'$ then there is $v \in W$ with vZv' and $uS_w v$.

First we note that if one considers quotients with respect to strong autobisimulations the problem with transitivity is solved and we may take definition as originally proposed by de Jonge, i.e. $\overline{u}\,\overline{S}_{\overline{w}}\,\overline{v}$ if and only if $(\exists x \in \overline{w})(\exists y \in \overline{u})(\exists z \in \overline{v})yS_x z$.

To see this, assume that $\overline{u}\,\overline{S}_{\overline{w}}\,\overline{v}\,\overline{S}_{\overline{w}}\,\overline{a}$. So there exist $x_1 \in \overline{w}, y \in \overline{u}, z_1 \in \overline{v}$ with $yS_{x_1}z_1$ and $x_2 \in \overline{w}, z_2 \in \overline{v}, b \in \overline{a}$ with $z_2 S_{x_2} b$. Now we have $x_1 Z x_2$, $z_1 Z z_2$, $x_1 R z_1$ and $z_2 S_{x_2} b$. The $(back2)$ property implies that there exists c such that $z_1 S_{x_1} c$ and cZb. Now we have $yS_{x_1}z_1 S_{x_1} c$, so by transitivity $yS_{x_1}c$. We conclude that $\overline{u}\,\overline{S}_{\overline{w}}\,\overline{a}$.

It was proved in [3] that every strong bisimulation is a bisimulation and that the quotient mapping from W to $\overline{W}$ if a bisimulation, provided that the starting autobisimulation Z is a strong, reflexive, transitive and symmetric bisimulation.

Now we give a slightly stronger version of the latter result.

Proposition 4. *If W is a Veltman model and Z a strong autobisimulation on W that is also an equivalence relation then the quotient mapping $\pi(w) = \overline{w}$ is a strong bisimulation between W and $\overline{W}$.*

Proof. Conditions (at), $(forth1)$ and $(back1)$ are easy to check and the condition $(forth2)$ is trivial. For $(back2)$ assume that we have $w\pi\overline{w}$, $u\pi\overline{u}$, wRu and $\overline{u}\,\overline{S}_{\overline{w}}\,\overline{v}$. So there exist $x \in \overline{w}, y \in \overline{u}$ and $z \in \overline{v}$ with $yS_x z$.

Now we have wZx, uZy, wRu and $yS_x z$. By the $(back2)$ property of

Z there exists a with uS_wa and aZz. Thus uS_wa and $a\pi\overline{v}$, so π satisfies $(back2)$. $\square$

Now we prove that when we consider two Veltman models, their quotient structures determine global bisimulations between them.

First we need the notion of isomorphism between two Veltman models.

Definition 5. If $W = \langle W, R, \{S_w : w \in W\}, \Vdash \rangle$ and $W' = \langle W', R', \{S'_{w'} : w' \in W'\}, \Vdash \rangle$ are two Veltman models an isomorphism between W and W' is a bijective map $f : W \to W'$ that satisfies the following conditions:

- $\forall w \in W(w \Vdash p \Leftrightarrow f(w) \Vdash p)$, for all propositional letters p,

- $\forall w, u \in W(wRu \Leftrightarrow f(w)R'f(u))$,

- $\forall w, u, v \in W(uS_wv \Leftrightarrow f(u)S'_{f(w)}f(v))$.

Proposition 6. *Let W_1, W_2 be two Veltman models and Z_1, Z_2 strong autobisimulations on those models that are also equivalence relations. If the quotient structures $\overline{W_1}$ and $\overline{W_2}$ (with respect to Z_1 and Z_2) are isomorphic, then W_1 and W_2 are globally bisimilar.*

Proof. Let $f : \overline{W_1} \to \overline{W_2}$ be an isomorphism of quotient models. We define relation X with $w_1 X w_2$ if and only if $w_2 \in f(\overline{w_1})$. We now prove that X is a global bisimulation between W_1 and W_2.

To see that X is a global relation, take $w_1 \in W_1$. As Z_2 is reflexive, we know that $f(\overline{w_1})$ is nonempty, thus there exists some $w_2 \in W_2$ with $w_1 X w_2$. Conversely, if $w_2 \in W_2$, then there is some $\overline{w} \in \overline{W_1}$ with $f(\overline{w}) = \overline{w_2}$. Take any $w_1 \in \overline{w}$. As $\overline{w_1} = \overline{w}$, we have $f(\overline{w_1}) = \overline{w_2}$, thus $w_1 X w_2$.

Now we show that X is a strong bisimulation.

(at) Assume $w_1 X w_2$. Now we have $w_1 \Vdash p$ iff $\overline{w_1} \Vdash p$ iff $f(\overline{w_1}) \Vdash p$ iff $\overline{w_2} \Vdash p$ iff $w_2 \Vdash p$. The first and the last equivalence hold by definition. The second equivalence is true because f is an isomorphism and the third because $\overline{w_2} = f(\overline{w_1})$.

$(forth1)$ Let $w_1 X w_2$ and $w_1 R_1 u_1$. Now we have $\overline{w_1}\overline{R_1}\overline{w_2}$ and, as f is an isomorphism, we get $f(\overline{w_1})\overline{R_2}f(\overline{u_1})$ and thus $\overline{w_2}\overline{R_2}f(\overline{u_1})$. This implies that there exist $x \in \overline{w_2}$ and $y \in f(\overline{v_1})$ with xR_2y. Now we have w_2Z_2x and xR_2y. The property $(back1)$ of Z_2 implies that there is some $u_2 \in W_2$ with $w_2R_2u_2$ and u_2Z_2y. Thus $\overline{u_2} = f(\overline{u_1})$, so $u_1 X u_2$.

($forth2$) Let w_1Xw_2, u_1Xu_2, $w_2R_2u_2$ and $u_1S^1_{w_1}v_1$. Then $\overline{u_1}\,\overline{S^1}_{\overline{w_1}}\overline{v_1}$ and as f is an isomorphism $f(\overline{u_1})\overline{S^2}_{f(\overline{w_1})}f(\overline{v_1})$, and as $f(\overline{w_1}) = \overline{w_2}$ and $f(\overline{u_1}) = \overline{u_2}$ we get $\overline{u_2}\,\overline{S^2}_{\overline{w_2}}f(\overline{v_1})$.

So there exist $x \in \overline{w_2}, y \in \overline{u_2}$ and $z \in f(\overline{v_1})$ with yS^2_xz. Now we have w_2Z_2x, u_2Z_2y, $w_2R_2u_2$ and yS^2_xz. By the ($back2$) property of Z_2 there exists some v_2 with $u_2S^2_{w_2}v_2$ and v_2Z_2z. Thus $u_2S^2_{w_2}v_2$ and v_1Xv_2, so the condition ($forth2$) is satisfied.

Conditions ($back1$) and ($back2$) can be proven similarly.   □

Reverse of the previous proposition is true with some assumptions on bisimulations Z_1 and Z_2.

Proposition 7. *Let W_1 and W_2 be two Veltman models and Z a strong global bisimulation between them. If Z_1 and Z_2 are strong autobisimulations (on W_1 and W_2 respectively) that are also the largest (ordinary) bisimulations on W_1 and W_2 respectively, then the quotients $\overline{W_1}$ and $\overline{W_2}$(with respect to Z_1 and Z_2) are isomorphic.*

Proof. It is easy to check that the largest bisimulation on W_1 (and W_2) is an equivalence relation. We define $f : \overline{W_1} \to \overline{W_2}$ with

$$f(\overline{w_1}) = \overline{w_2} \text{ iff } (\exists x_1 \in \overline{w_1})(\exists x_2 \in \overline{w_2})x_1Zx_2.$$

To see that f is well-defined assume that we have $u_1, v_1 \in W_1$ and $u_2, v_2 \in W_2$ with u_1Zu_2, v_1Zv_2 and $u_1Z_1v_1$. Then we have $u_2(Z^{-1} \circ Z_1 \circ Z)v_2$. As a composition of bisimulations is a bisimulation and Z_2 is the largest bisimulation on W_2, we have $Z^{-1} \circ Z_1 \circ Z \subseteq Z_2$ and thus $u_2Z_2v_2$, so f is well-defined.

Now we prove that f if an isomorphism.

To show that f is onto, assume $w_2 \in W_2$. As Z is a global bisimulation, there exists $w_1 \in W_1$ with w_1Zw_2. So by definition $f(\overline{w_1}) = \overline{w_2}$.

For injectivity, assume that we have $u_1, v_1 \in W_1$ with $f(\overline{u_1}) = f(\overline{v_1})$. From the definition of f, there exist $x_1 \in \overline{u_1}$ and $x_2 \in f(\overline{v_1})$ with x_1Zx_2. It follows that $u_1Z_1x_1$ and $\overline{x_2} = f(\overline{v_1})$. Again, there exist $y_1 \in \overline{v_1}$ and $y_2 \in \overline{x_2}$ with y_1Zy_2. Now, as we have $y_1Z_1v_1$, $y_2Z_2x_2$, it follows that $u_1(Z_1 \circ Z \circ Z_2 \circ Z^{-1} \circ Z_1)v_1$. As $Z_1 \circ Z \circ Z_2 \circ Z^{-1} \circ Z_1$ is a bisimulation on W_1, and Z_1 is the largest one, we have u_1Zv_1. It follows that f is injective.

The first condition in definition 5 is trivial to prove.

To show the second condition, we need to prove that

$$\overline{u_1}\,\overline{R_1}\,\overline{v_1} \text{ iff } f(\overline{u_1})\overline{R_2}f(\overline{v_1}).$$

Assume first that $\overline{u_1}\,\overline{R_1}\,\overline{v_1}$. From the definition of $\overline{R_1}$ it follows that there exist $x_1 \in \overline{u_1}$ and $y_1 \in \overline{v_1}$ with $x_1 R_1 y_1$. As Z is a global relation, there exists $x_2 \in W_2$ with $x_1 Z x_2$. Now, by the condition $(forth1)$ of Z, there exists $y_2 \in W_2$ with $x_2 R_2 y_2$ and $y_1 Z y_2$. From $x_1 \in \overline{u_1}$ and $x_1 Z x_2$ we get $\overline{x_2} = f(\overline{u_1})$ and similarly $\overline{y_2} = f(\overline{v_1})$. Thus $f(\overline{u_1})\overline{R_2}f(\overline{v_1})$, as desired.

Conversely, assume that $f(\overline{u_1})\overline{R_2}f(\overline{v_1})$. By definition, there exist $x_2 \in f(\overline{u_1})$ and $y_2 \in f(\overline{v_1})$ with $x_2 R_2 y_2$. As Z is a global relation, we have $x_1 Z x_2$ for some $x_1 \in W_1$. Using the $(back1)$ condition of Z, there is some $y_1 \in W_1$ with $x_1 R_1 y_1$ and $y_1 Z y_2$. As $\overline{x_2} = f(\overline{u_1})$, there exist $z_1 \in \overline{u_1}$ and $z_2 \in \overline{x_2}$ with $z_1 Z z_2$. Summing up, we have $x_1 Z x_2$, $x_2 Z_2 z_2$, $z_1 Z z_2$ and $z_1 Z_1 u_1$. Thus $x_1(Z \circ Z_2 \circ Z^{-1} \circ Z_1)u_1$, and as Z_1 is the largest bisimulation on W_1 we get $x_1 Z_1 u_1$. Similarly we prove that $y_1 Z_1 v_1$. We conclude that $\overline{u_1}\,\overline{R_1}\,\overline{v_1}$, as desired.

We still have to prove that $\overline{u_1}\,\overline{S^1}_{\overline{w_1}}\,\overline{v_1}$ if and only if $f(\overline{u_1})\overline{S^2}_{f(\overline{w_1})}f(\overline{v_1})$.

Let us first suppose that we have $\overline{u_1}\,\overline{S^1}_{\overline{w_1}}\,\overline{v_1}$. Then there exist $x_1 \in \overline{w_1}$, $y_1 \in \overline{u_1}$ and $z_1 \in \overline{v_1}$ with $y_1 S^1_{x_1} z_1$. As Z is global, there exists $x_2 \in W_2$ with $x_1 Z x_2$. Now using the $(forth1)$ condition for Z there exists $y_2 \in W_2$ with $y_1 Z y_2$ and $x_2 R_2 y_2$.

We have $x_1 Z x_2$, $y_1 Z y_1$, $x_2 R_2 y_2$ and $y_1 S^1_{x_1} z_1$, so by the $(forth2)$ condition for Z there exists $z_2 \in W_2$ with $y_2 S^2_{x_2} z_2$ and $z_1 Z z_2$. From the definition of the quotient structure $\overline{W_2}$ we get $\overline{y_2}\,\overline{S^2}_{\overline{x_2}}\,\overline{z_2}$ and as $\overline{x_2} = f(\overline{x_1}) = f(\overline{w_1})$, $\overline{y_2} = f(\overline{y_1}) = f(\overline{u_1})$ and $\overline{z_2} = f(\overline{z_1}) = f(\overline{v_1})$ we get $f(\overline{u_1})\overline{S^2}_{f(\overline{w_1})}f(\overline{v_1})$.

For the other direction we first assume that we have $f(\overline{u_1})\overline{S^2}_{f(\overline{w_1})}f(\overline{v_1})$. Then there exist $x_2 \in f(\overline{w_1})$, $y_2 \in f(\overline{u_1})$ and $z_2 \in f(\overline{v_1})$ with $y_2 S^2_{x_2} z_2$. As Z is global, there exists $x_1 \in W_1$ with $x_1 Z x_2$. By the $(back1)$ condition for Z we get $y_1 \in W_1$ with $x_1 R_1 y_1$ and $y_1 Z y_2$. Now we have $x_1 Z x_2$, $y_1 Z y_2$, $x_1 R_1 y1$ and $y_2 S^2_{x_2} z_2$, so by the $(back2)$ condition for Z there exists $z_1 \in W_1$ with $y_1 S^1_{x_1} z_1$ and $z_1 Z z_2$. As $\overline{x_2} = f(\overline{w_1})$, there exist $k_1 \in \overline{w_1}$ and $k_2 \in \overline{x_2}$ with $k_1 Z k_2$. We have $x_1 Z x_2, x_2 Z_2 k_2, k_1 Z k_2$ and $k_1 Z_1 w_1$, thus $x_1(Z \circ Z_2 \circ Z^{-1} \circ Z_1)w_1$. As Z_1 is the largest bisimulation on W_1, we get $x_1 Z_1 w_1$, thus $\overline{x_1} = \overline{w_1}$. Similarly we get $\overline{y_1} = \overline{u_1}$ and $\overline{z_1} = \overline{v_1}$, so we have $\overline{u_1}\,\overline{S^1}_{\overline{w_1}}\,\overline{v_1}$ as desired. $\square$

From the previous results one might conclude that strong bisimulation

is the correct notion of bisimulation, at least when quotient structures are considered. However, strong bisimulations have certain undesired properties. For example, it is hard to determine the largest strong autobisimulation of a model, as the union of two strong bisimulations is not necessarily a strong bisimulation. In fact, starting with a strong bisimulation, one might not even be able to construct another strong bisimulation containing it that would be reflexive, as we now show.
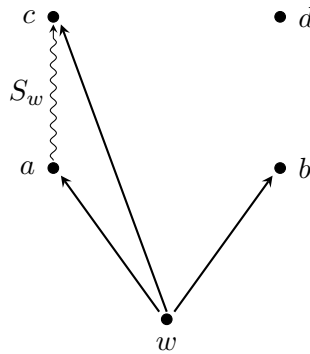
Consider the following Veltman model:

$$W = \{w, a, b, c, d\}$$

$$R = \{(w, a), (w, b), (w, c)\}$$

$$S_w = \{(a, c)\}$$

Here we assume that the conditions implied by the definition of Veltman model are satisfied. By picturing we get:


It is easy to check that $Z = \{(a, b), (c, d)\}$ and $Z_0 = \{(x, x) : x \in W\}$ are two strong bisimulations on W. However, their union is not a strong bisimulation, as we have wZ_0w, aZb, wRb and aS_wc, but there is no x with $c(Z \cup Z_0)x$ and bS_wx.

¿From the previous example we know that the diagonal Z_0 is always a strong autobisimulation. One can also easily see that if $Z \subseteq W \times W'$ is a strong bisimulation, then so is it's inverse Z^{-1}. As we have seen,

the union of strong autobisimulations does not necessarily yield a strong autobisimulation. It is the same with composition, as we now show.

Consider Veltman models W, W' and W'' as bellow:

$W:$

$W':$
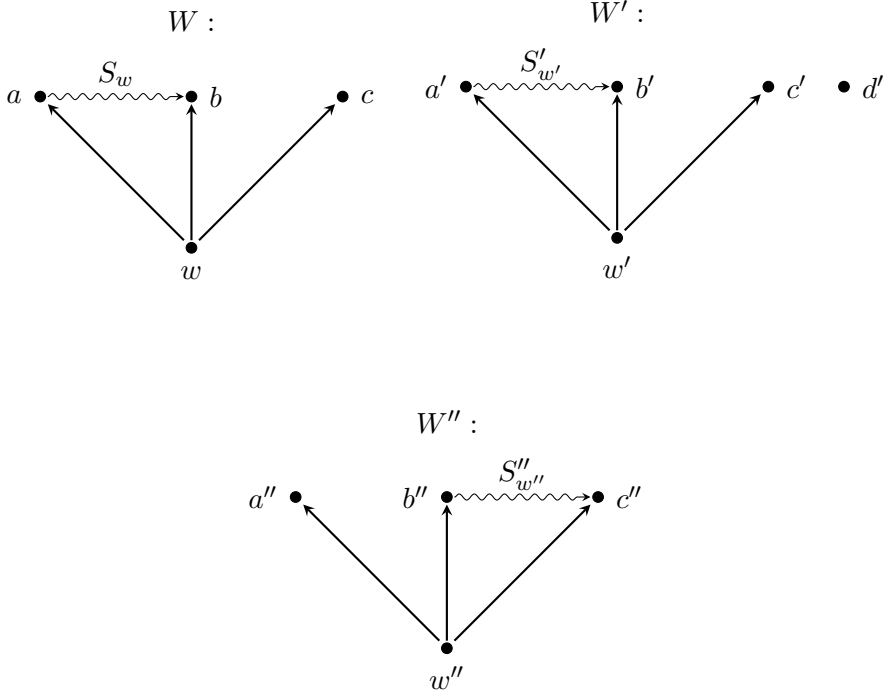
It is easy to see that $Z = \{(w, w'), (a, d'), (a, a'), (b, b'), (c, c')\}$ and $Z' = \{(w', w''), (d', a''), (a', b''), (c', a''), (b', c'')\}$ are strong bisimulations.

However, we have $w(Z \circ Z')w''$, $a(Z \circ Z')a''$, $w''R''a''$ and aS_wb, but there is no x'' such that $a''S''_{w''}x''$ and $b(Z \circ Z')x''$.

By taking disjoint unions of the models in previous example(se e.g [3]) we get that even the composition of strong autobisimulations isn't necessarily a strong autobisimulation, so the standard way to find the largest autobisimulation can't be applied to search for the largest strong autobisimulation.

Note that in proposition 7 we use the assumption that strong autobisimulations Z_1 and Z_2 are also largest (ordinary)autobisimulations on respective models. Thus it would be nice if the largest autobisimulation was also a strong bisimulation. Unfortunately, this is not the case. We have already mentioned an example constructed by de Jonge in [3] where

the quotient mapping is not a bisimulation. On a closer inspection one can notice that in this example the autobisimulation used to form the quotient is in fact the largest autobisimulation, and, as strong bisimulations give quotients that are bisimilar to the starting model, this bisimulation, although the largest one, is not a strong bisimulation.

M. Vuković proved in [9] Hennessy–Milner theorem for generalized Veltman semantics, and he considered the existence of bisimulation between Veltman model and generalized Veltman model in [10]. We would like to define bisimulation quotients of generelized Veltman models and try to prove similar results.

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Faculty of Agriculture
University of Zagreb
Svetošimunska cesta 25
10000 Zagreb, CROATIA

domagojvrgoc@gmail.com

Department of Mathematics
University of Zagreb
Bijenička cesta 30
10000 Zagreb, CROATIA

vukovic@math.hr