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## A LATTICE OF IMPLICATIVE EXTENSIONS OF REGULAR KLEENE’S LOGICS

**A b s t r a c t.** The paper deals with functional properties of three-valued logics. We consider the family of regular three-valued Kleene’s logics (strong, weak, intermediate) and it’s extensions by adding an implicative connectives (“natural” implications). The main result of our paper is the lattice that describes the relations between implicative extensions of regular logics.

### 1. Introduction

In this paper we propose an original approach to a problem of relation between different three-valued logics. And the family of regular three-valued Kleene’s logics is considered as the base for other three-valued logics.

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In [10] S.C. Kleene represented two regular logics: a strong Kleene's logic  $\mathbf{K}_3$  and a weak Kleene's logic  $\mathbf{K}_3^w$ . Truth-tables for propositional connectives of these logics are regular in the following sense: the given column(row) contains 1 in the row(column) for  $\frac{1}{2}$  only iff the either column(row) consists of 1; similary for 0.

M. Fitting in [6] describes another one regular logic, which is intermediate between the weak and the strong Kleene's logics. Such logic is called **Lisp** ( $\mathbf{K}_3^\rightarrow$ ). Logic **Lisp** and relation between regular logics are particularly investigated in [11]. In this paper another one intermediate regular logic is presented. Such logic is called **Twin Lisp** and it is functional equivalent to **Lisp**. Along it's enough to consider one of them.

In our research we follow the functional treatment of the notion of logic and 3-valued logic is defined as some finite set of propositional connectives, established by truth-tables. This approach is very convenient for comparison of essentially different logics.

As regular logic we consider the logic of following type:  $\{\sim, \vee, \wedge\}$ , where  $\sim$  — regular negation,  $\vee, \wedge$  — regular disjunction and conjunction.

Strong Kleene's logic  $\mathbf{K}_3$  is  $\{\sim, \vee, \wedge\}$ , where  $\vee, \wedge$  are defined by the following strong regular tables<sup>1</sup>:

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

Weak Kleene's logic  $\mathbf{K}_3^w$  is  $\{\sim, \cup, \cap\}$ , where  $\cup, \cap$  are defined by the following weak regular tables:

$\cup$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\cap$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	0	$\frac{1}{2}$	0

Intermediate three-valued logic **Lisp**  $\mathbf{K}_3^\rightarrow$  is  $\{\sim, \vee^\rightarrow, \wedge^\rightarrow\}$ , where  $\vee^\rightarrow, \wedge^\rightarrow$  are defined by the following regular tables:

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<sup>1</sup>Note, that  $\sim$  defines equally in every three-valued regular logic:  $\sim 0 = 1, \sim 1 = 0, \sim \frac{1}{2} = \frac{1}{2}$ .

$\vee \rightarrow$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\wedge \rightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	0	0	0

It is notable that in any three-valued regular logic with one designated value 1, the set of tautology is empty. At all these systems implicative connective is not a primitive connective, but it can be defined by using  $\sim$  and  $\vee$  connectives:  $p \supset q =_{df} \sim p \vee q$ .

Of special interest is consideration of extensions of regular Kleene's logics by adding an implicative connective with some «good» properties. For example, we know that three-valued Bochvar's logic can be regarded as implicative extension of weak Kleene's logic, strong Kleene's logic with Jaśkowski implication is taken for construction of logic **PCont**. So, the object of our article is to consider systematically implicative extensions of all regular Kleene's logics and represent them as a lattice.

## 2. *Natural* implication

DEFINITION. Let  $V_3$  be a set of truth values  $\{0, \frac{1}{2}, 1\}$  and  $D$  be a set of designated values. Implication is called *natural* if it is satisfied the following criteria:

1. **C**-extending, i.e. restrictions to the subset  $\{0, 1\}$  of  $V_3$  coincide with the classical implication.
2. If  $x \rightarrow y \in D$  and  $x \in D$ , then  $y \in D$ , i.e. matrices for implication need to be normal in the sense of Łukasiewicz-Tarski (they verify the modus ponens) [13, p. 134].
3. Let  $x \leq y$ , then  $x \rightarrow y \in D$ .
4.  $x \rightarrow y \in V_3$ , in other cases.

According to the definition of *natural* implication, there are 6 implications with  $D = \{1\}$ :

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	$a$	0
$\frac{1}{2}$	1	1	$b$
0	1	1	1

where  $a \in \{0, \frac{1}{2}\}$  and  $b \in \{0, \frac{1}{2}, 1\}$ .

With  $D = \{1, \frac{1}{2}\}$  there are 24 implications:

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	$b$	0
$\frac{1}{2}$	$a$	$a$	0
0	1	$a$	1

where  $a \in \{1, \frac{1}{2}\}$  and  $b \in \{0, \frac{1}{2}, 1\}$ .

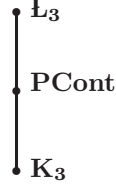
Notice, that 2 paires of implications are the same with  $D = \{1\}$  and  $D = \{1, \frac{1}{2}\}$ . So, the class of natural implications consists of 28 implications. Among them the implications of best-known three-valued logics: implication of Łukasiewicz logic [12], implication of Bochvar's logic **B<sub>3</sub>** [3], Jaśkowski implication (1948), which then appears in [2] and [15] (logic **PCont**), implication of Sobociński [16], which appears in logic **RM3** [1], implication of logic of Heyting (1930) and standard Resher's implication [14], implication of paraconsistent logic of Sette **P<sub>1</sub>** [17] and others. All these logics are analysed in detail in [9].

Next we investigate the implicative extensions by *natural* implication of all regular Kleene's logics (**K<sub>3</sub>**, **K<sub>3</sub><sup>→</sup>**, **K<sub>3</sub><sup>w</sup>**).

### 3. Implicative extensions of Kleene's logics

On examination of implicative extensions of strong Kleene's logic, we have received 2 classes of logics: the class of systems which are functional equivalent to Łukasiewicz's logic **L<sub>3</sub>** and the class of systems which are functional equivalent to logic **PCont**.

Notice that extension of logic **PCont** by constant 1 leads to well-known paraconsistent logic **J<sub>3</sub>**, which is functionally equivalent to logic **L<sub>3</sub>**. So, the extensions of **K<sub>3</sub>** form a simply ordered set:



On examination of implicative extensions of intermediate Kleene's logic, we have received 3 classes of logics: the class of systems which are functional equivalent to Łukasiewicz's logic  $\mathbf{L}_3$ ; the class of systems which are functional equivalent to logic  $\mathbf{PCont}$  and the class of systems which are functional equivalent to logic  $\mathbf{T}^2$ . Logic  $\mathbf{T}^2$  is not previously considered in literature.

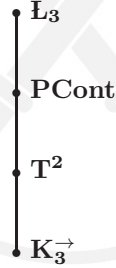
$$\mathbf{T}^2 = \mathbf{K}_3^{\rightarrow} + \rightarrow_i \ (i \in \{23, 24\})$$

The implications  $\rightarrow_{23}$  and  $\rightarrow_{24}$  satisfy the criteria of *natural* implication and their truth-tables are as follows:

$\rightarrow_{23}$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	1	$\frac{1}{2}$	1

$\rightarrow_{24}$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	1	$\frac{1}{2}$	1

The extensions of  $\mathbf{K}_3^{\rightarrow}$  form the following simply ordered set:



On examination of implicative extensions of weak Kleene's logic, we received 7 *basic* logics: Łukasiewicz's logic  $\mathbf{L}_3$ , paraconsistent logic  $\mathbf{PCont}$ , three-valued Bochvar's logic  $\mathbf{B}_3$ , logic  $\mathbf{Z}$  [7],  $\mathbf{T}^3$ ,  $\mathbf{T}^2$  and  $\mathbf{T}^1$ .

$$\mathbf{T}^1 = \mathbf{K}_3^w + \rightarrow_{23}$$

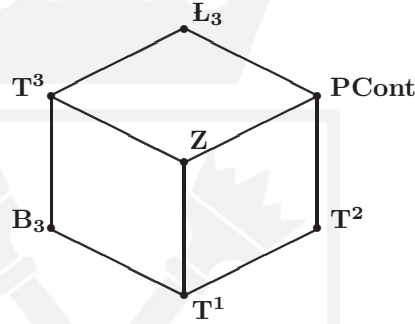
$$\mathbf{T}^2 = \mathbf{K}_3^w + \rightarrow_{24}$$

$$\mathbf{T}^3 = \mathbf{K}_3^w + \rightarrow_{13}$$

Logics  $\mathbf{T}^1$  and  $\mathbf{T}^3$  are also first described. Implication  $\rightarrow_{13}$  satisfies the criteria of *natural* implication and it is represented by the following table:

$\rightarrow_{13}$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	0
0	1	$\frac{1}{2}$	1

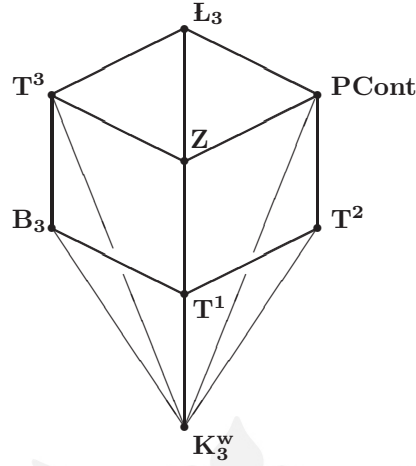
So, 7 basic logics form a lattice w.r.t. relation of functional inclusion one logic to another.



It is important to note that logics  $\mathbf{T}^1$ ,  $\mathbf{T}^2$  and  $\mathbf{T}^3$  (which are not before presented in literature) are *non-commutative*. It is obvious if we define disjunctions  $\mathbf{T}^1$ ,  $\mathbf{T}^2$  and  $\mathbf{T}^3$  in the standard way by using  $\sim$  and  $\rightarrow_i$  ( $i \in \{23, 24, 13\}$ ).

From a functional standpoint logic  $\mathbf{T}^1$  is the weakest extension of logic  $\mathbf{K}_3^w$ . In paper [5] 11 sets of functions precomplete in  $\mathbf{B}_3$  (in the set of functions that correspond to the logic  $\mathbf{B}_3$ ) are described. And logic  $\mathbf{T}^1$  is one of these sets, namely  $\mathbf{T}^1$  corresponds to the set of internal functions.

But this bring up the question, why logic of Halldén  $\mathbf{H}_3$  [8] and logic of Ebbinghaus  $\mathbf{E}_3$  [4] are not appeared in our systematization, in spite of the fact that these systems can also be regarded as the extensions of logic  $\mathbf{K}_3^w$ . Let's consider the lattice  $\mathbf{L}(\mathbf{K}_3^w)$ :

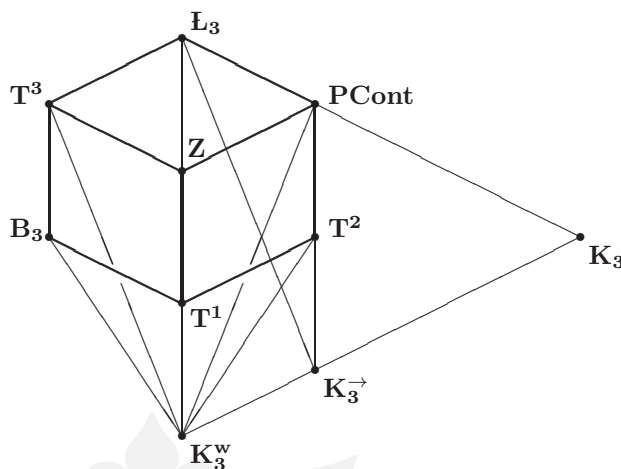


So, logic  $\mathbf{H}_3$  is absent in our systematization, because none of *natural* implications cannot be defined in  $\mathbf{H}_3$ . In other words, logic  $\mathbf{H}_3$  is not an implicative extension of weak Kleene's logic. It is known that this system is between weak Kleene's logic  $\mathbf{K}_3^w$  and Bochvar's logic  $\mathbf{B}_3$ .

Logic of Ebbinghaus  $\mathbf{E}_3$  is not appeared in our systematization, because we cannot receive this system right from the weak Kleene's logic by adding one of the natural implications; it's necessary to extend  $\mathbf{K}_3^w$  to  $\mathbf{B}_3$ , and then from  $\mathbf{B}_3$  we receive (by adding any one of the implications, which we use for obtaining logic  $\mathbf{Z}$  from  $\mathbf{K}_3^w$ )  $\mathbf{E}_3$ . (Or we can first extend  $\mathbf{K}_3^w$  to  $\mathbf{Z}$  and then  $\mathbf{E}_3$  can be received from  $\mathbf{Z}$  by adding any one of the implications, which we use for obtaining logic  $\mathbf{B}_3$  from  $\mathbf{K}_3^w$ .) Logic  $\mathbf{E}_3$  is between  $\mathbf{B}_3$  and  $\mathbf{L}_3$  on the one part,  $\mathbf{Z}$  and  $\mathbf{T}^3$  on the other part. Analogously we can receive logic  $\mathbf{T}^4$  from logic  $\mathbf{T}^2$  by it's extension by any one of the implications, which we use for obtaining logic  $\mathbf{Z}$  from  $\mathbf{K}_3^w$  (or from logic  $\mathbf{Z}$  by it's extension by any one of the implications, which we use for obtaining logic  $\mathbf{T}^2$  from  $\mathbf{K}_3^w$ ). Exactly as  $\mathbf{E}_3$  logic  $\mathbf{T}^4$  is not an implicative extension of weak Kleene's logic.

Let's summarize the results<sup>2</sup> obtained during our investigation and present implicative extensions of all regular logics —  $\mathbf{K}_3$ ,  $\mathbf{K}_3^{\rightarrow}$  and  $\mathbf{K}_3^w$  in the following way:

<sup>2</sup>The proofs of obtained results are given in [18].



Hence, we can use any regular logic as a base for construction of  $\mathbf{L_3}$  and  $\mathbf{PCont}$ , logic  $\mathbf{T^2}$  can be constructed as implicative extensions of intermediate Kleene's logic ( $\mathbf{K_3^{\rightarrow}}$ ) or weak Kleene's logic ( $\mathbf{K_3^w}$ ), logics  $\mathbf{B_3}$ ,  $\mathbf{Z}$ ,  $\mathbf{T^1}$  and  $\mathbf{T^3}$  are appeared exceptionally as implicative extensions of  $\mathbf{K_3^w}$ .

$$K. p \rightarrow (q \rightarrow p)$$

$$S. (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)).$$

Thus our approach allows us to separate out different classes of equivalent constructions for different 3-valued logics. Moreover presented lattice structures visually demonstrate the relationships between different 3-valued logics.

## References

- [1] A.R. Anderson and N.D. Belnap, *Entailment: The Logic of Relevance and Necessity*, Princeton University Press (1975).
- [2] D. Batens, *Paraconsistent extensional propositional logics*, *Logique et Analyse* **23**:90–91 (1980), pp. 127–139.



- [3] D.A. Bochvar, *On a three-valued logical calculus and its applicaion ti the analysis of the paradoxes f the classical functional calculus*, History and Philosophy of Logic **2** (1938/1981), pp. 87–112.
- [4] H.-G. Ebbinghaus, *Über eine pradikatenlogik mit partiell definierten Pradikaten und Funktionen*, Arch. Math. Logik Grundlagenforsch **12** (1969), pp. 39–53.
- [5] V.K. Finn, *A Criterion of Functional Completeness for  $B_3$* , Studia Logica **33**:2 (1974), pp. 121–125.
- [6] M. Fitting, *Kleene's three valued logics and their children*, Fundamenta Informaticae **20** (1992), pp. 113–131.
- [7] K. Hałkowska, *A note on matrices for systems of nonsens-logic*, Studia Logica **48**:4 (1989), pp. 461–464.
- [8] S. Hallden, *The Logic of Nonsense*, Uppsala, 1949.
- [9] A.S. Karpenko, *The Development of Many-Valued Logic*, LKI Pblishers, Moscow, 2010 (in Russian).
- [10] S.C. Kleene, *Introduction to Metamatematics*, D. Van Nostrand, Princeton, NJ, 1952.
- [11] E.Y. Komendantskaya, *Functional interdependence of regular Kleene logics*, Logical Investigations **15** (2009), Nauka, Moscow, pp. 116–128 (in Russian).
- [12] J. Łukasiewicz, *O logice trójwartościowej*, Ruch Filozoficzny **5** (1920), pp. 170–171. (English translation: *On three-valued logic*, Łukasiewicz J. Selected Works, North-Holland & PWN, Amsterdam & Warszawa 1970.)
- [13] J. Łukasiewicz, A. Tarski, *Untersuchungen über den Aussagenkalkul*, Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie, III(23) (1930), pp. 1–21. (English translation: *Investigations into the sentential calculus*, Łukasiewicz J. Selected Works, North-Holland & PWN Amsterdam & Warszawa, 1970.)
- [14] N. Resher, *Many-valued Logic*, McGraw-Hill, New York 1969.
- [15] L.I. Rozonoer, *On contradictions discovering in formal theories. I*, Automation and Remote Control **6** (1983), pp. 113–124 (in Russian).
- [16] B. Sobociński, *Axiomatization of a partial system of three-valued calculus of propositions*, The Journal of Computing Systems **11**:1 (1952), pp. 23–55.
- [17] A.M. Sette, *On propositional calculus P1*, Mathematica Japonica **16** (1973), pp. 173–180.
- [18] N. Tomova, *Regular Kleene's logics: extension and generalization*, Ph.D. dissertation, Lomonosov Moscow State University, 2010 (in Russian).

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