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# ULTRAFILTERS (WITH DENSE ELEMENTS) OVER CLOSURE SPACES

A b s t r a c t. Several notions and results that are useful for directed sets (and their applications) can be extended to the more general context of closure spaces; inter alia the so-called "finite intersection property" and the existence of special ultrafilters (namely ultrafilters which elements are dense) on such structures.

## 1. Introduction

Our source of inspiration is the case of the directed sets, that have been studied for their topological applications ([5], [6], [7]), but showed also interesting links with infinite combinatorics, large cardinals and ultrafilters ([2], [3], [4], [5], [6], [7]).
A careful analysis shows that some of the involved notions and results concern essentially the implicit "closure space" aspect, and that this allows

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generalizations.
The notion of "breakpoint" for example is present in several fields, but has not (as far as we know) been studied in a unified way; the same can be said for the notion of "measurability" (using ultrafilters). We study this here in a rather large context, namely the one of "closure spaces", and show the links relating those matters with the "finite intersection property".
At the end we discuss the case of the partial orders.

## 2. Basic definitions and facts

**Definition 2.1.** A "closure space" is a non-empty set $E$ provided with a "closure operator" $C\ell : \mathcal{P}E \to \mathcal{P}E$ (where $\mathcal{P}E$ is the powerset of $E$) satisfying the following conditions :

1. $X \subset C\ell\ X$
2. $C\ell\ C\ell\ X = C\ell\ X$
3. $X \subset Y \to C\ell\ X \subset C\ell Y$

**Definition 2.2.** A closure space $E$ is "acyclic" iff $\forall x \in E$, $C\ell\{x\} \neq E$.

This terminology is inspired by the one used for groups; any group can be seen as a "closure space" for the operator "subgroup generated by".

**Definition 2.3.** A subset $X$ of a closure space $E$ is

- "dense" iff $C\ell\ X = E$
- "strictly dense" iff $C\ell\ X = E$ and $C\ell(E \setminus X) \neq E$
- "closed" iff $C\ell\ X = X$

Notice that the empty set $\emptyset$ is never dense in an acyclic closure space $E$.

**Definition 2.4.** A partition $\mathcal{F}$ of an acyclic closure space $E$ is "fatal" iff no element $X$ of $\mathcal{F}$ is dense in $E$.

**Remark.** We will de facto only be interested in acyclic closure spaces; and for these the existence of a "fatal" partition is obviously guaranteed.

**Definition 2.5.** The "breakpoint" of an acyclic closure space $E$ is the least possible cardinal for a fatal partition of $E$ (notation : $\delta_E$).

**Fact 2.6.** *$\delta_E$ is exactly the least possible cardinal of a covering of $E$ by proper closed subsets.*

**Proof.** Let $C = (X_\alpha)_{\alpha<\theta}$ be a covering of space $E$ by proper closed subsets. We define inductively a sequence $(Y_\alpha)_{\alpha<\theta}$ by the formula $Y_\alpha = X_\alpha \setminus U\{X_\beta : \beta < \alpha\}$. Then $\{Y_\alpha : \alpha < \theta\}$ is a partition of $E$ into non-dense sets. $\square$

**Fact 2.7.** *For an acyclic closure space $E$ :*

$$2 \leq \delta_E \leq |E|.$$

**Fact 2.8.** *For each cardinal $\delta \geq 2$, there exists an acyclic closure space $M$ such that $\delta_M = \delta$.*

**Proof.** Take (f.ex.) $M = \delta$, with

$$\begin{cases} C\ell\ X = M & \text{iff} \quad |X| \geq 2 \\ C\ell\ X = X & \text{iff} \quad |X| < 2 \end{cases}$$

Notice that this shows that the breakpoint is not necessarily a regular cardinal. $\square$

## 3. Some examples

**3.1.** Any topological space is obviously a closure space, satisfying the extra condition that finite unions of closed subsets are still closed. Notice that the breakpoint $\delta_E$ is not necessarily exactly the additivity $\kappa_E :=$ the largest cardinal $\kappa$ such that any $\kappa$-finite intersection of open sets is an open set.

**Counter-example:** $\mathbb{R}$ with the usual topology; here $\delta = 2$, while $\kappa = \aleph_0$.
All that can be said in general is that $\delta_E \leq \kappa_E$ (easy proof).

**3.2.** A partial order $(E, \leq)$ is a closure space for the "downards closure" operator :

$$(X] := \{y \in E | \exists x \in X \quad y \leq x\}.$$

$E$ is then at the same time a topological space and an "algebraic" closure space (i.e. one where $C\ell X = \cup\{C\ell\ Y | Y \subset X \text{ and } Y \text{ is finite}\}$).
Notice that the condition of "being acyclic" corresponds here to the fact that there is no maximum element in $E$.

**3.3.** For a non-empty directed set $D$ without maximum element, with $C\ell\ X := (X]$, the breakpoint $\delta_D$ is exactly the "characteristic" ([2], [3], [5], [6]), and is necessarily a regular cardinal.

**3.4.** A group $(G, \cdot)$ is a closure space for $C\ell\ X :=$ the subgroup generated by $X$; and $G$ is then "acyclic" iff $G$ is not a "cyclic group".
One can easily show that $\delta_G > 2$ for any group $G$. In the case of $(\mathbb{Z} \times \mathbb{Z}, +)$ for example, the breakpoint is 3 : just consider the 3 subgroups respectively defined by :

- the elements which first component is even
- the elements which second component is even
- the elements having both components of same parity.

Other example : the breakpoint of $(\mathbb{Q}, +)$ is $\aleph_0$.

**3.5.** A vector space is a closure space for $C\ell\ X :=$ the subspace generated by $X$ ("linear span"). In the case of $\mathbb{R}^3$ for example, the breakpoint is $2^{\aleph_0}$.

Indeed : the set of all 1-dimensional subspaces is a covering of cardinality $2^{\aleph_0}$. So the breakpoint is at most $2^{\aleph_0}$.
Further can any covering by proper subspaces be reduced to a minimal list of distinct elements, all of dimension 1 or 2, and such that no element of dimension 1 is ever included in an element of dimension 2; and such a non-redundant list is obviously at least of cardinality $2^{\aleph_0}$.

**3.6.** In the order structure $(\mathbb{N}, \leq)$ with $C\ell X := (X]$, the dense subsets are the "cofinal" ones, and the "strictly dense subsets" are the "cofinite" ones.

## 4. Families of dense subsets

**Notation 4.1.** *For an acyclic closure space $E$, we will be particularly interested in several properties for the following families :*

$$\begin{aligned} \mathcal{D}(E) &= \{X | X \subset E \text{ and } X \text{ is dense in } E\} \\ \mathcal{SD}(E) &= \{X | X \subset E \text{ and } X \text{ is strictly dense in } E\} \end{aligned}$$

**Definition 4.2.** A family $\mathcal{F}$ of subsets of $E$ has the "FIP" ("finite intersection property"; see [1]) iff any finite intersection of elements of $\mathcal{F}$ is non-empty ("finite intersection" : "intersection of finitely many").

**Properties 4.3.** *Let $E$ be an acyclic space. Then :*

**4.3.1.** If $A \subset B \subset E$ and $A \in \mathcal{SD}(E)$, then $B \in \mathcal{SD}(E)$

**4.3.2.** If $A \in \mathcal{SD}(E)$ and $B \in \mathcal{SD}(E)$, then $A \cap B \neq \emptyset$

**4.3.3.** If $\delta_E \geq 4$ and $A, B \in \mathcal{SD}(E)$, then $A \cap B \in \mathcal{D}(E)$

**4.3.4.** If $\delta_E \geq \aleph_0$, then any finite intersection of elements of $\mathcal{SD}(E)$ belongs to $\mathcal{D}(E)$; a fortiori : $\mathcal{SD}(E)$ has the FIP

**4.3.5.** If $2 < \delta_E < \aleph_0$, then $\mathcal{SD}(E)$ does not have the FIP.
Hint : consider a fatal partition $\{A_0, A_1, A_2, \ldots, A_{\delta-1}\}$; then $\bigcap_{i<\delta}(E \setminus A_i) = \emptyset$, with each $E \setminus A_i$ strictly dense.

**Remark 4.4.** Property 4.3.3 is "optimal" in the sense that for $\delta_E = 2$ and $\delta_E = 3$, the implication does not necessarily hold.

**Counter-examples :**

1. $E = \{1, 2, 3, 4\}$, with

$$ClX := \begin{cases} X & \text{if} \quad |X| \leq 2 \\ E & \text{if} \quad |X| \geq 3 \end{cases}$$

Here $\delta_E = 2$, both $\{1, 2, 3\}$ and $\{2, 3, 4\}$ are strictly dense, but $\{1, 2, 3\} \cap \{2, 3, 4\}$ is not dense.

2. $E = \{1,2,3\}$, with

$$ClX := \begin{cases} X & \text{if} \quad |X| \leq 1 \\ E & \text{if} \quad |X| \geq 2 \end{cases}$$

Here $\delta_E = 3$, both $\{1,2\}$ and $\{2,3\}$ are strictly dense, but $\{1,2\} \cap \{2,3\}$ is not dense.

**Conclusion 4.5.** *For any acyclic closure space $E$ :*

1. $\delta_E \geq \aleph_0 \to \mathcal{SD}(E)$ *has the FIP*
2. *($\delta_E < \aleph_0$ and $\mathcal{SD}(E)$ has the FIP) $\to \delta_E = 2$*

**Remark 4.6.** $\mathcal{D}(E)$ seldom has the FIP, even when $\delta_E$ is infinite (cf. the examples in section 3).

## 5. Particular ultrafilters on acyclic closure spaces

Let us recall that any family having the FIP can be extended to an ultrafilter (see e.g. [1]). Therefore, $SD(E)$ has FIP if and only if it is contained in some ultrafilter.
We discuss this here for the family $\mathcal{SD}(E)$, with $E$ an acyclic closure space.

**Fact 5.1.** *If $\delta_E > 2$ and $\mathcal{SD}(E) \subset \mathcal{U}$ (an ultrafilter), then $\mathcal{U} \subset \mathcal{D}(E)$.*
**Proof.** By the hypotheses, $\mathcal{SD}(E)$ should have the FIP; by Property 4.3.5 : $\delta_E \geq \aleph_0$.
So take $u \in \mathcal{U}$ and suppose $u \notin \mathcal{D}(E)$.
Now $\{u, E \setminus u\}$ is a partition of $E$, and cannot be fatal (because $\delta_E > 2$).
So $E \setminus u$ should be dense, and even strictly dense (because we supposed $u \notin \mathcal{D}(E)$). But then $E \setminus u \in \mathcal{U}$ : a contradiction. □

**Fact 5.2.** *If (an ultrafilter) $\mathcal{U} \subset \mathcal{D}(E)$, then $\mathcal{SD}(E) \subset \mathcal{U}$ and $\delta_E \geq \aleph_0$.*
**Proof.** Consider a strictly dense subset $X$ of $E$. If $X \notin \mathcal{U}$, then $E \setminus X \in \mathcal{U}$, and so (by our hypothesis) $E \setminus X$ is dense, contradicting the definition of "strictly dense" (for $X$).
So $\mathcal{SD}(E) \subset \mathcal{U}$, and $\mathcal{SD}(E)$ has the FIP, excluding $3 \leq \delta_E < \aleph_0$ (by property 4.3.5). The situation $\delta_E = 2$ is also excluded, because for any

$Y \subset E$, necessarily $Y$ or $E \setminus Y$ is in $\mathcal{U}$, so that no partition into two pieces can ever be fatal.

**Conclusion :** $\delta_E \geq \aleph_0$. □

**Fact 5.3.** *If $\delta_E \geq \aleph_0$, then any ultrafilter $\mathcal{U}$ realizing $\mathcal{SD}(E) \subset \mathcal{U}$ or $\mathcal{U} \subset \mathcal{D}(E)$ realizes these both conditions, and is necessarily non-principal.*

**Proof.** Combine facts 5.1 and 5.2, and remember that no singleton can be dense in $E$ (acyclic). □

**Fact 5.4.** *If $\delta_E < \aleph_0$, then no ultrafilter $\mathcal{U}$ can realize $\mathcal{U} \subset \mathcal{D}(E)$.*

**Proof.** If $\delta_E < \aleph_0$ and $\mathcal{U} \subset \mathcal{D}(E)$, there exists a finite fatal partition; one piece should be in $\mathcal{U}$, so should be dense, contradicting the definition of "fatal" partition. □

**Synthesis 5.5** (for $E$ acyclic closure space).

1. when $\delta_E$ is infinite, there do exist ultrafilters $\mathcal{U}$ such that $\mathcal{SD}(E) \subset \mathcal{U} \subset \mathcal{D}(E)$; these $\mathcal{U}$ are necessarily non-principal;
2. when $\delta_E$ is finite but $> 2$, no ultrafilter $\mathcal{U}$ can ever realize $\mathcal{U} \subset \mathcal{D}(E)$, nor $\mathcal{SD}(E) \subset \mathcal{U}$;
3. when $\delta_E = 2$, no ultrafilter $\mathcal{U}$ can ever realize $\mathcal{U} \subset \mathcal{D}(E)$.

**Remark.** In case 3., the existence of an ultrafilter $\mathcal{U}$ such that $\mathcal{SD}(E) \subset \mathcal{U}$ is not necessarily excluded; section 7 provides related examples.

**5.6. Remarks about $\delta_E$-complete ultrafilters.**

As usually, "$\delta_E$-complete" means "closed under $\delta_E$-finite intersections". It is easy to see that, when $\delta_E$ is infinite, $\mathcal{SD}(E)$ has the $\delta_E$-FIP (i.e. any $\delta_E$-finite intersection of elements of $\mathcal{SD}(E)$ is non-empty), but has not the $\delta_E^+$-FIP ($\delta_E$ being the successor cardinal of $\delta$). So that, if some ultrafilter $\mathcal{U}$ extends $\mathcal{SD}(E)$ (or, equivalently, is included in $\mathcal{D}(E)$) and is $\delta_E$-complete, necessarily this $\mathcal{U}$ cannot be $\delta_E^+$-complete. Then by the classical result [1], proposition 4.2.7, $\delta_E$ has to be a measurable cardinal. On the other hand, if $\delta_E$ is strongly compact (i.e. any $\delta_E$-complete filter can be extended to a $\delta_E$-complete ultrafilter), obviously $\mathcal{SD}(E)$ can be extended to a $\delta_E$-complete ultrafilter.

## 6. Application : the case of the ordered sets

We explore here the class of the ordered sets without maximum element seen as (acyclic) closure spaces, via the closure operator $Cl(X) := (X]$ (see 3.2).

### 6.1. Case where $E$ is directed

Then the breakpoint $\delta_E$ is a regular (infinite) cardinal (see 3.3), so that $\mathcal{SD}(E)$ does have the FIP (see 4.5) and there exist ultrafilters $\mathcal{U}$ such that $\mathcal{SD}(E) \subset \mathcal{U} \subset \mathcal{D}(E)$. Remember that the dense subsets are the cofinal ones, in this context.
The concerned ultrafilters are necessarily non-principal.

### 6.2. Case were $E$ is not directed

Then $\delta_E$ is necessarily 2 : take $a, b$ such that $\{a, b\}$ has no upper bound in $E$; then $\{[a), E \setminus [a)\}$ is a fatal partition. Notice also that no ultrafilter $\mathcal{U}$ on $E$ can realize $\mathcal{U} \subset \mathcal{D}(E)$ (fact 5.4).
What can be said about ultrafilters $\mathcal{U}$ realizing $\mathcal{SD}(E) \subset \mathcal{U}$ ?
That question is closely related to the presence of maximal elements in $E$.

**Theorem 6.2.1.** *If $E$ has no maximal elements, then $\mathcal{SD}(E)$ has not the FIP (and so $\mathcal{SD}(E) \subset \mathcal{U}$ (ultrafilter) is excluded).*

**Proof.**[1]

**Step 1.** There exist $A, B$, disjoint dense subsets of $E$.
**Step 2.** As $E$ is not directed, some pair $\{x, y\} \subset E$ has no (upper) bound. Then nor $[x)$, nor $E \setminus [x)$ can be dense in $E$.
**Step 3.** Consider $A, B$ as in step 1, and $x, y$ as in step 2. Now the sets $A \cup [x)$, $B \cup [x)$, $A \cup (E \setminus [x))$, $B \cup (E \setminus [x))$ are all strictly dense because they extend dense subsets, and their complements are respectively parts of $E \setminus [x)$, $E \setminus [x)$, $[x)$, $[x)$, which are not dense.
But the intersection of the 4 so constructed strictly dense subsets of $E$ is obviously empty. □

### 6.2.2 Case where $E$ is not directed, but admits maximal elements.

Let $M$ be the (non-empty) set of the maximal elements of $E$. Then necessarily $\mathcal{SD}(E)$ does have the FIP, as any dense subset $X$ will contain

[1]This proof is due to Armin Rigo and Olivier Esser

$M$. So in any case $\mathcal{SD}(E)$ can be extended into an ultrafilter $\mathcal{U}$. Can one hope to have non-principal ones ? The answer is given by

**Theorem 6.2.3.** *There exists a non-principal ultrafilter $\mathcal{SD}(E) \subset \mathcal{U}$ iff $M$ is infinite.*

**Proof.**

- Suppose $M$ $(\neq \emptyset)$ is finite and $\mathcal{SD}(E) \subset \mathcal{U}$, a non-principal ultrafilter. Consider $E' := E \setminus (M]$.
  If $E' = \emptyset$, then $M$ is strictly dense, so $M \in \mathcal{U}$ : a contradiction.
  If $E' \neq \emptyset$, construct (as in step 1 of the proof of theorem 6.2.1) disjoint sets $A$ and $B$, both dense in $E'$. Now $A \cup M$ and $B \cup M$ are obviously strictly dense in $E$, so should belong to $\mathcal{U}$. But their intersection is exactly $M$ (finite) : a contradiction.
- Suppose $M$ is infinite. Just put some non-principal ultrafilter $\mathcal{U}'$ on $M$, and define $\mathcal{U} := \{X | X \subset E \text{ and } X \cap M \in \mathcal{U}'\}$.
  Then $\mathcal{U}$ is obviously a non-principal ultrafilter on $E$. Further, if $X$ is dense in $E$, necessarily $M \subset X$, so that $\mathcal{SD}(E) \subset \mathcal{U}$. □

**6.2.4.** In the proof of theorems 6.2.1 and 6.2.3 we used disjoint dense subset $A$ and $B$ : their existence is guaranteed in any partial order without maximal elements, but not necessarily for general acyclic closure spaces !

**Counter-example :** $E =$ the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, with addition modulo 2, and $C\ell(X) :=$ the subgroup generated by $X$. Here the dense subsets are exactly : $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \{1\}$, $\{1\} \times \mathbb{Z}_2$, which are not pairwise disjoint.

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