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RELATIONAL AND NEIGHBORHOOD SEMANTICS FOR INTUITIONISTIC MODAL LOGIC

A b s t r a c t. We investigate semantics for an intuitionistic modal logic in which the "possibility" modality does not distribute over disjunction. In particular, the main aim of this paper is to study such intuitionistic modal logic as a variant of classical non-normal modal logic. We first give a neighborhood semantics together with a sound and complete axiomatization. Next, we study relationships between our approach and the relational (Kripke-style) semantics considered in the literature. It is shown that a relational model can be represented as a neighborhood model, and the converse direction holds under a slight restriction. Also, by considering degenerate cases of neighborhood and relational semantics, we demonstrate that a certain classical monotone modal logic has relational semantics, and can be embedded into a classical normal bimodal logic.

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1. Introduction

1.1 Background

Intuitionistic modal logics that do not admit the "distributivity" law $\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$ have been considered from several motivations in the literature. For example, Wijesekera considered such a logic in view of concurrent dynamic logic [15]. Fairtlough and Mendler introduced the "lax modality" in view of the application to hardware verification [6]. Kobayashi considered a constructive S4 in the context of typed lambda-calculus [9], and Kripke semantics for the same logic has been investigated by Alechina, Mendler, de Paiva and Ritter [1]. Hilken's theory of Stone duality for intuitionistic modal algebras also treats a $\Diamond$ operator without distributivity [8].

Semantics for intuitionistic modal logic has been considered mainly in Kripke-style, but there is a difficulty in such an approach: the usual interpretation of $\Diamond$ (and $\vee$) validates the distributivity. This is roughly because existential quantification, which is used to interpret $\Diamond$ in the meta-level, distributes over disjunction. So, to avoid the distributivity, we need to fix the interpretation of $\Diamond$ [15, 1]. As a result of such a modification, the analogy between intuitionistic and classical modal logic is lost; this makes intuitionistic modal logics harder to understand as a variant of classical modal logics. For example, consider the correspondences between axioms and properties of frames. One of the simplest example is the axiom $p \to \Diamond p$: classically this axiom corresponds to reflexivity, but in modified Kripke semantics this would not hold.

In classical setting, it is known that a modality $\Diamond$ without distributivity cannot be handled in the usual Kripke semantics, and such modalities is said to be *non-normal*. One of the alternative tools to study such a modality is neighborhood semantics [11]. However, its intuitionistic version has not been extensively studied. Although Sotirov [14] and Wijesekera [15] considers neighborhood semantics for intuitionistic modal logic, it does not seem that they tried to capture the nature of non-distributive $\Diamond$ in terms of neighborhood semantics. In Sotirov's work only a necessity modality is considered, and Wijesekera's semantics requires some extra axioms for completeness.

In this paper, we will investigate intuitionistic modal logic without distributivity as a *non-normal* modal logic. This point of view has not been considered before; in preceding researches, $\Diamond$ without distributivity in intuitionistic setting has not been referred to as non-normal. This paper demonstrates that the neighborhood-style approach, which has been developed to capture classical non-normal modal logics, is also applicable in the intuitionistic setting.

1.2 Overview

In this paper, we will consider neighborhood semantics for intuitionistic modal logic (which differs from Sotirov's or Wijesekera's ones). We will discuss

1. the relationship between existing relational (Kripke-style) semantics and our neighborhood semantics, and

2. the classical case of our framework, and the relationship with classical monotone and bimodal logics.

Relational and neighborhood semantics for intuitionistic modal logic are basically obtained by adding a preorder (taken from ordinary Kripke semantics for intuitionistic logic) to the corresponding classical semantics. So a relational frame is a triple $\langle W, \leq, R \rangle$, where R is a binary relation, and a neighborhood frame is a triple $\langle W, \leq, N \rangle$, where N is a neighborhood function, which is a mapping from W to $\mathcal{P}(\mathcal{P}(W))$.

For 1, we show that relational and neighborhood semantics are "almost" equivalent. This is done by defining mutual translations between relational and neighborhood models. Precisely speaking, not all neighborhood models have relational representation. What we actually do is to define "normal" neighborhood models, and show that each normal neighborhood model can be transformed into an equivalent relational model. The converse direction is easier: any relational model can be transformed into an equivalent normal neighborhood model. As an immediate consequence of these translations, we can see that relational semantics and normal neighborhood semantics define the same logic.

For 2, we will consider a certain classical monotone modal logic, and show that it has relational semantics (although it is not in the scope of the usual Kripke semantics), and it can be embedded into $\mathrm{S5} \otimes \mathrm{K}$, a classical normal bimodal logic with S5 and K modalities. First, we will observe that a classical neighborhood model for monotone modal logic can be regarded as a special case of our neighborhood model. This is done by regarding a classical neighborhood frame $\langle W, N\rangle$ as $\langle W, =, N\rangle$. In other words, a classical neighborhood frame is just an intuitionistic one whose $\leq$-part is degenerate. Under this identification, we apply a translation given in 1. to classical neighborhood models. This derives a relational representation of classical monotone modal logic. Next, we use the fact that a relational model is also a model of $\mathrm{S5} \otimes \mathrm{K}$, which is easy to verify. This observation, together with the relational representation, induces a translation from classical monotone modal logic to $\mathrm{S5} \otimes \mathrm{K}$.

1.3 Organization of the Paper

In Section 2 we will review existing approaches to intuitionistic modal logics. After listing some basic definitions, we introduce relational semantics and a sound and complete axiomatization.

Section 3 introduces neighborhood semantics. We first define a neighborhood semantics, and give a sound and complete axiomatization. The resulting logic is slightly weaker than the logic introduced in Section 2. After seeing this, we will define normal neighborhood models, and show that they determine the same logic as the relational semantics.

In Section 4 we will give translations between relational and normal neighborhood models, and show that these translations do not change the interpretation of formulas in an appropriate sense.

In Section 5, we will consider classical modal logic. We first introduce a neighborhood semantics for a classical monotone modal logic. This logic turns out to be a classical variant of the logic considered in Section 3. We will define relational semantics for classical monotone modal logic, and establish an embedding into a classical bimodal logic $\mathrm{S5} \otimes \mathrm{K}$.

Finally, in Section 6 we summarize the paper, discuss related work and make some remarks on informal interpretation of our results.

1.4 Basic Settings and Notations

Throughout this paper, we use the following notations:

- PV is a fixed infinite set of propositional variables, and ranged over by p, q;
- $\mathbf{L}(O_1, \dots, O_n)$ is the set of formulas built from PV and $\bot$ with logical connectives $\wedge, \vee, \to$ and unary modalities $O_1, \dots, O_n$;
- $\neg A$ is an abbreviation of $A \to \bot$.

Mostly we will consider $\mathbf{L}(\Box, \Diamond)$ in this paper.

Definition 1.1. 1. A set $\Lambda \subseteq \mathbf{L}(\Box, \Diamond)$ is said to be an $\mathbf{L}(\Box, \Diamond)$-*logic* if it contains all intuitionistic tautologies, and closed under uniform substitution, modus ponens, and necessitation (that is, if $A \in \Lambda$, then $\Box A \in \Lambda$).

2. Let Λ be an $\mathbf{L}(\Box, \Diamond)$-logic. A set $\Lambda' \subseteq \mathbf{L}(\Box, \Diamond)$ is said to be a Λ-*logic* if Λ' is a $\mathbf{L}(\Box, \Diamond)$-logic and $\Lambda \subseteq \Lambda'$.

3. Let Λ be an $\mathbf{L}(\Box, \Diamond)$-logic. A set $\Gamma \subseteq \mathbf{L}(\Box, \Diamond)$ is said to be a Λ-*theory* if it contains Λ and closed under modus ponens.

The following are axioms that will appear in this paper.

($N_{\Diamond\Box}$)	$\Diamond\bot \to \Box\bot$	(K)	$\Box(p \to q) \to \Box p \to \Box q$
($N_{\Diamond}$)	$\neg\Diamond\bot$	($K_{\Diamond}$)	$\Box(p \to q) \to \Diamond p \to \Diamond q$
(PEM)	$p \vee \neg p$		

Definition 1.2. The $\mathbf{L}(\Box, \Diamond)$-logic IM is the least logic containing K and $K_{\Diamond}$

Definition 1.3. Let A be a formula, and Λ be an IM-logic. We define $\Lambda + A$ as the smallest Λ-logic containing A.

2. Relational Semantics

2.1 Existing Relational Approaches

There have been several approaches to define a Kripke-style semantics for intuitionistic modal logic. Most of them are obtained by introducing an extra structure into the Kripke semantics for intuitionistic logic. Below we will consider a triple $\langle W, \leq, R\rangle$.[1] Here $\leq$ and R are taken from the Kripke semantics for intuitionistic logic and modal logic, respectively.

A problem in introducing modalities in this way is that the ordinary truth conditions for modalities

$$x \Vdash \Box A \iff \forall y.\ (x\ R\ y \implies y \Vdash A),$$
$$x \Vdash \Diamond A \iff \exists y.\ (x\ R\ y \text{ and } y \Vdash A)$$

breaks the heredity condition

$$x \leq y \text{ and } x \Vdash A \implies y \Vdash A,$$

which is expected in the Kripke semantics for intuitionistic logic.

Several solutions to this problem have been proposed in the literature. Roughly speaking, there are two approaches:

1. consider an alternative truth condition, and
2. impose some conditions on $\leq$ and R,

for each of $\Box$ and $\Diamond$. For example, Plotkin and Stirling [12] consider 1. for $\Box$ and 2. for $\Diamond$. This approach results in a logic which admits the distributivity of $\Diamond$ over disjunction. Wolter and Zakharyaschev [16] consider 2. for $\Box$ and 1. for $\Diamond$. In this approach, although the distributivity can be rejected, the duality $\Diamond A \leftrightarrow \neg\Box\neg A$ becomes a theorem. Since this is not necessarily natural in intuitionistic setting, we take another approach; we will consider 2. for both $\Box$ and $\Diamond$. We will define

$$x \Vdash \Box A \iff \forall z \geq x.\ \forall y.\ (z\ R\ y \implies y \Vdash A),$$
$$x \Vdash \Diamond A \iff \forall z \geq x.\ \exists y.\ (z\ R\ y \text{ and } y \Vdash A).$$

This is the choice often taken in the previous literature to model a $\Diamond$ without distributivity [15, 6, 1].

The above truth condition for $\Diamond$ may look strange, because it breaks the analogy between $\Diamond$ and $\exists$. Actually, a $\Diamond$ modality without distributivity

[1] Another approach often seen in the literature is to consider first-order Kripke structure, and interpret $\Box$ and $\Diamond$ as $\forall$ and $\exists$ in it. This approach has been considered by Ewald [5] and Simpson [13], for example.

cannot be interpreted as $\exists$, because $\exists x.(P(x) \vee Q(x))$ implies $(\exists x.P(x)) \vee (\exists x.Q(x))$ in intuitionistic first-order logic.

Below, we will summarize known results on the relational semantics for intuitionistic modal logic based on this approach. Basically the contents of the rest of this section is a propositional fragment of Wijesekera's work [15].

2.2 Definition of Relational Semantics

Definition 2.1. An *intuitionistic relational frame* is a triple $\langle W, \leq, R\rangle$ of a non-empty set W, a preorder $\leq$ on W, and a binary relation R on W.

Definition 2.2. 1. For an intuitionistic relational frame $\mathcal{R} = \langle W, \leq, R\rangle$, an $\mathcal{R}$-*valuation* is a map V from PV to $\mathcal{P}(W)$.

2. An $\mathcal{R}$-valuation V is said to be *admissible* if $V(p)$ is upward-closed for all propositional variables p.

Definition 2.3. An *intuitionistic relational model* is a pair $\langle \mathcal{R}, V\rangle$ of an intuitionistic relational frame $\mathcal{R}$ and an admissible $\mathcal{R}$-valuation V.

Definition 2.4. Let $\langle \mathcal{R}, V\rangle$ be an intuitionistic relational model. We can define the satisfaction relation, denoted by $\Vdash_{\mathrm{r}}$, as follows:

$$\begin{aligned}
\mathcal{R}, V, x \Vdash_{\mathrm{r}} p &\iff x \in V(p);\\
\mathcal{R}, V, x \Vdash_{\mathrm{r}} A \wedge B &\iff \mathcal{R}, V, x \Vdash_{\mathrm{r}} A \text{ and } \mathcal{R}, V, x \Vdash_{\mathrm{r}} B;\\
\mathcal{R}, V, x \Vdash_{\mathrm{r}} A \vee B &\iff \mathcal{R}, V, x \Vdash_{\mathrm{r}} A \text{ or } \mathcal{R}, V, x \Vdash_{\mathrm{r}} B;\\
\mathcal{R}, V, x \Vdash_{\mathrm{r}} A \to B &\iff \forall y \geq x.(\mathcal{R}, V, y \Vdash_{\mathrm{r}} A \implies \mathcal{R}, V, y \Vdash_{\mathrm{r}} B);\\
\mathcal{R}, V, x \Vdash_{\mathrm{r}} \Box A &\iff \forall y \geq x.\forall z.(y\ R\ z \implies \mathcal{R}, V, z \Vdash_{\mathrm{r}} A);\\
\mathcal{R}, V, x \Vdash_{\mathrm{r}} \Diamond A &\iff \forall y \geq x.\exists z.(y\ R\ z \text{ and } \mathcal{R}, V, z \Vdash_{\mathrm{r}} A).
\end{aligned}$$

Below we sometimes suppress $\mathcal{R}$ and V if they are clear from the context.

It is easy to verify that the heredity condition

$$\mathcal{R}, V, x \Vdash_{\mathrm{r}} A \text{ and } x \leq y \implies \mathcal{R}, V, y \Vdash_{\mathrm{r}} A$$

holds for all formulas $A \in \mathbf{L}(\Box, \Diamond)$.

Definition 2.5. Let A be a formula in $\mathbf{L}(\Box, \Diamond)$.

1. Let $\langle \mathcal{R}, V\rangle$ be an intuitionistic relational model. A is said to be *true in* $\langle \mathcal{R}, V\rangle$ if $\mathcal{R}, V, x \Vdash_{\mathrm{r}} A$ for all $x \in W$.

2. Let $\mathcal{K}$ be a class of intuitionistic relational models. A is said to be *true in* $\mathcal{K}$ if it is true in all models of $\mathcal{K}$.

Theorem 2.6 (Soundness and Completeness)**.** *$A \in \mathbf{L}(\Box, \Diamond)$ is a theorem of* $\mathrm{IM} + \mathrm{N}_{\Diamond}$ *if and only if A is true in all intuitionistic relational models.*

Soundness is proved by routine induction, and the completeness can be proved by canonical model construction (we omit the details). Another proof using completeness for neighborhood semantics can be found at the end of Section 4.

3. Neighborhood Semantics

3.1 Definition of Neighborhood Semantics

Classically a neighborhood frame is given by a pair $\langle W, N\rangle$, where W is a set of possible worlds and N is a map from W to $\mathcal{P}(\mathcal{P}(W))$[4, Part III]. We call N a neighborhood function. Here we consider its intuitionistic version, so we introduce additional relation $\leq$ to model intuitionistic behavior.

Definition 3.1. An *intuitionistic neighborhood frame* is a triple $\langle W, \leq, N\rangle$ of a non-empty set W, a preorder $\leq$ on W, and a mapping $N : W \to \mathcal{P}(\mathcal{P}(W))$ that satisfies the *decreasing condition*:

$$x \leq y \implies N(x) \supseteq N(y).$$

The notion of valuation is defined in the same way as the relational case.

Definition 3.2. For an intuitionistic neighborhood frame $\mathcal{N} = \langle W, \leq, N\rangle$, an *$\mathcal{N}$-valuation* is a map V from PV to $\mathcal{P}(W)$. An $\mathcal{N}$-valuation V is said to be *admissible* if $V(p)$ is upward-closed for all $p \in \mathrm{PV}$.

Definition 3.3. An *intuitionistic neighborhood model* is a pair $\langle \mathcal{N}, V \rangle$ of an intuitionistic neighborhood frame $\mathcal{N}$ and an admissible $\mathcal{N}$-valuation V.

Definition 3.4. Given an intuitionistic neighborhood model $\langle \mathcal{N}, V \rangle$, we can define the satisfaction relation, denoted by $\Vdash_{\mathrm{n}}$, in the same way as in the relational case, except that the truth conditions of modalities read

$$\begin{aligned} \mathcal{N}, V, x \Vdash_{\mathrm{n}} \Box A &\iff \forall X \in N(x). \forall y \in X. \mathcal{N}, V, y \Vdash_{\mathrm{n}} A; \\ \mathcal{N}, V, x \Vdash_{\mathrm{n}} \Diamond A &\iff \forall X \in N(x). \exists y \in X. \mathcal{N}, V, y \Vdash_{\mathrm{n}} A. \end{aligned} \tag{1}$$

The notion of truth in a model and a class of models is defined in the same way as Definition 2.5.

Remark 3.5. The conditions (1) are different from the ones in the usual neighborhood semantics, which read

$$\begin{aligned} \mathcal{N}, V, x \Vdash_{\mathrm{n}} \Box A &\iff \exists X \in N(x). \forall y. (y \in X \iff \mathcal{N}, V, y \Vdash_{\mathrm{n}} A); \\ \mathcal{N}, V, x \Vdash_{\mathrm{n}} \Diamond A &\iff \forall X \in N(x). \exists y. \neg(y \notin X \iff \mathcal{N}, V, y \Vdash_{\mathrm{n}} A). \end{aligned} \tag{2}$$

This difference is motivated from our goal, that is, to establish a model of modal logic with normal $\Box$ and non-normal $\Diamond$. In particular, unlike the usual classical modal logic, $\Box$ and $\Diamond$ cannot be each other's dual.

3.2 Sound and Complete Axiomatization

The proof system presented in the previous section is not sound for the neighborhood semantics introduced above, because the axiom $\mathrm{N}_\Diamond$ ($\Diamond\bot \to \bot$) is not necessarily true in all intuitionistic neighborhood models (for a counterexample, see Lemma 3.8). However, we can establish a sound and complete axiomatization by slightly modifying this axiom.

Theorem 3.6 (Soundness and Completeness). *$A \in \mathbf{L}(\Box, \Diamond)$ is a theorem of $\mathrm{IM} + \mathrm{N}_{\Diamond\Box}$ if and only if A is true in all intuitionistic neighborhood models.*

Proof. "Only if" part is proved in the usual way. For the converse direction, see Appendix A. $\Box$

3.3 Normal Worlds

Although the logic determined by the neighborhood semantics introduced above does not coincide with the logic of relational semantics, we can find a class of intuitionistic neighborhood frames that characterizes the logic of relational semantics.

First, we introduce the notion of normal worlds, at which the axiom $N_\Diamond$ holds.

Definition 3.7. Let $\mathcal{N} = \langle W, \leq, N\rangle$ be an intuitionistic neighborhood frame.

1. A possible world $x \in W$ is said to be *normal* if $N(x) \neq \emptyset$.
2. $\mathcal{N}$ is said to be *normal* if every world $x \in W$ is normal.
3. An intuitionistic neighborhood model $\langle \mathcal{N}, V\rangle$ is said to be *normal* if $\mathcal{N}$ is normal.

The main observation is the following lemma.

Lemma 3.8. *Let $\langle \mathcal{N}, V\rangle$ be an intuitionistic neighborhood model. Then, a world $x \in W$ is normal if and only if it satisfies $\mathcal{N}, V, x \Vdash_{\mathrm{n}} \neg\Diamond\bot$ (not depending on the choice of V).*

Theorem 3.9. *A formula A is a theorem of* $\mathrm{IM} + \mathrm{N}_\Diamond$ *if and only if A is true in all normal intuitionistic neighborhood models.*

Proof. "Only if" part is verified from the previous lemma and Theorem 3.6. For the converse direction, see Appendix A. □

4. Translations Between the Two Semantics

In the previous section, we have developed a neighborhood semantics for intuitionistic modal logic, and observed that normal neighborhood frames correspond to the logic determined by relational semantics. In this section, we will give mutual translations between relational models and normal neighborhood models.

We first consider a translation from relational to normal neighborhood models, and then we consider the converse direction. Both of the translations are proved to "preserve semantics" in an appropriate sense. These translations explicitly relate relational and neighborhood approach.

4.1 From Relational to Neighborhood Semantics

Definition 4.1. Let $\mathcal{R} = \langle W, \leq, R \rangle$ be an intuitionistic relational frame. Then we define the intuitionistic neighborhood frame induced from $\mathcal{R}$ to be the tuple $\mathcal{N}_{\mathcal{R}} = \langle W, \leq, N_R \rangle$, where

$$N_R(x) = \{R\,[y] \mid y \geq x\}\,, \text{ where } R\,[y] = \{z \mid y\ R\ z\}.$$

It is easy to check that $\mathcal{N}_{\mathcal{R}}$ is indeed an intuitionistic neighborhood frame for any intuitionistic relational frame $\mathcal{R}$. Additionally, an $\mathcal{R}$-valuation and an $\mathcal{N}_{\mathcal{R}}$-valuation are the same thing since they are both a mapping from PV to $\mathcal{P}(W)$. Admissibility of these valuations also coincide, since $\mathcal{R}$ and $\mathcal{N}_{\mathcal{R}}$ has the same preorder structure. To summarize, the following lemma holds:

Lemma 4.2. *Let $\langle \mathcal{R}, V \rangle$ be an intuitionistic relational model. Then $\langle \mathcal{N}_{\mathcal{R}}, V \rangle$ is a normal intuitionistic neighborhood model.*

Theorem 4.3. *Let $\langle \mathcal{R}, V \rangle$ be an intuitionistic relational model. Then,*

$$\mathcal{R}, V, x \Vdash_{\mathrm{r}} A \iff \mathcal{N}_{\mathcal{R}}, V, x \Vdash_{\mathrm{n}} A.$$

Proof. By induction on A. □

4.2 From Neighborhood to Relational Semantics

Definition 4.4. Let $\mathcal{N} = \langle W, \leq, N \rangle$ be an intuitionistic neighborhood frame. Then we define an intuitionistic relational frame induced from $\mathcal{N}$ as a tuple $\mathcal{R}_{\mathcal{N}} = \langle W^*, \leq^*, \ni^* \rangle$, where

$$\begin{aligned} &W^* = \{(x, X) \mid x \in W, X \in N(x)\}\,; \\ &(x, X) \leq^* (y, Y) \iff x \leq y; \\ &(x, X) \ni^* (y, Y) \iff y \in X. \end{aligned} \tag{3}$$

It is clear that $\mathcal{R}_{\mathcal{N}}$ is an intuitionistic relational frame for any intuitionistic neighborhood frame $\mathcal{N}$. This time, unlike the previous case, we need a little more consideration on valuations, since the set of possible worlds has been changed.

Definition 4.5. Let $\mathcal{N}$ be an intuitionistic neighborhood frame, and V an $\mathcal{N}$-valuation. Then we define an $\mathcal{R}_{\mathcal{N}}$-valuation V^* by

$$V^*(p) = \{(x, X) \in W^* \mid x \in V(p)\}.$$

Lemma 4.6. *Let $\mathcal{N}$ be an intuitionistic neighborhood frame, and V an $\mathcal{N}$-valuation. If V is admissible, then V^* is admissible. Therefore, if $\langle \mathcal{N}, V \rangle$ is an intuitionistic neighborhood model, then $\langle \mathcal{R}_{\mathcal{N}}, V^* \rangle$ is an intuitionistic relational model.*

If $\mathcal{N}$ is normal, this transformation preserves semantics in the following sense:

Theorem 4.7. *Let $\langle \mathcal{N}, V \rangle$ be a normal intuitionistic neighborhood model. Then the following are equivalent:*

1. $\mathcal{N}, V, x \Vdash_{\mathrm{n}} A$;
2. $\mathcal{R}_{\mathcal{N}}, V^*, (x, X) \Vdash_{\mathrm{r}} A$ *for all* $X \in N(x)$;
3. $\mathcal{R}_{\mathcal{N}}, V^*, (x, X) \Vdash_{\mathrm{r}} A$ *for some* $X \in N(x)$.

Proof. By induction on A. $\square$

Now, Theorem 2.6 can be proved as follows.

Proof of Theorem 2.6. By Theorem 3.9, a formula $A \in \mathbf{L}(\Box, \Diamond)$ is a theorem of $\mathrm{IM} + \mathrm{N}_{\Diamond}$ if and only if A is valid in all normal intuitionistic neighborhood models. So it suffices to see that A is true in all intuitionistic relational models if and only if A is true in all normal intuitionistic neighborhood models. "If" part follows from Theorem 4.3, and "only if" part follows from Theorem 4.7. $\square$

5. Application to the Classical Case

In the previous section, we have investigated the relationship between relational and neighborhood semantics. In this section, we apply the translation given above to the classical setting. As a result, we can obtain a relational representation of neighborhood semantics for classical monotone modal logics. Also, as a consequence of this representation, we show that there is an embedding from classical monotone modal logic $\mathrm{IM}+\mathrm{PEM}+\mathrm{N}_{\Diamond}$ to a certain classical bimodal logic.

5.1 Classical Monotone Modalities

First of all, we briefly introduce classical modal logic with both normal and non-normal modalities. The formulation here is basically taken from the course note by Pacuit [11].

We consider the language $\mathbf{L}([\,], [\,\rangle, \langle\,], \langle\,\rangle)$.

Definition 5.1. A *classical neighborhood frame* is a pair $\langle W, N\rangle$, where W is a non-empty set and N is a map from W to $\mathcal{P}(\mathcal{P}(W))$ (a neighborhood function).

Definition 5.2. For a classical neighborhood frame $\mathcal{N} = \langle W, N\rangle$, an $\mathcal{N}$*-valuation* is a map from PV to $\mathcal{P}(W)$. A classical neighborhood model is a pair $\langle \mathcal{N}, V\rangle$ of a classical neighborhood frame $\mathcal{N}$ and an $\mathcal{N}$-valuation V.

In a similar way to the intuitionistic case, we can define the truth of formulas by the following clauses:

$$\begin{aligned}
\mathcal{N}, V, x \Vdash [\,]A &\iff \forall X \in N(x).\forall y \in X.\mathcal{N}, V, y \Vdash A;\\
\mathcal{N}, V, x \Vdash [\,\rangle A &\iff \forall X \in N(x).\exists y \in X.\mathcal{N}, V, y \Vdash A;\\
\mathcal{N}, V, x \Vdash \langle\,]A &\iff \exists X \in N(x).\forall y \in X.\mathcal{N}, V, y \Vdash A;\\
\mathcal{N}, V, x \Vdash \langle\,\rangle A &\iff \exists X \in N(x).\exists y \in X.\mathcal{N}, V, y \Vdash A.
\end{aligned}$$

Propositional connectives are interpreted in the same way as the usual classical modal logic.

Remark 5.3. 1. $\langle\,]$ and $[\,\rangle$ can be regarded as $\Box$ and $\Diamond$ in classical monotone modal logics. So the $\mathbf{L}(\langle\,], [\,\rangle)$-fragment of the logic above is essentially the classical monotone modal logic, often called M or EM[4, Section 8.2].

2. $[\,]$ and $\langle\,\rangle$ are $\Box$ and $\Diamond$ in normal modal logic, that is, they satisfy the following axioms:

$$[\,](p \to p), \quad [\,](p \to q) \to [\,]p \to [\,]q,$$
$$\neg\langle\,\rangle\bot, \quad \langle\,\rangle(p \vee q) \to \langle\,\rangle p \vee \langle\,\rangle q.$$

Therefore, the $\mathbf{L}([\,], \langle\,\rangle)$-fragment of the logic above is the minimal normal modal logic K.

5.2 Relational Semantics for the Classical Setting

The classical modal logic defined above can be regarded as a special case of the neighborhood semantics defined in Section 3. This fact, combined with the result of Section 4, suggests that we can define a relational semantics for a monotone modal logic. Below we will see how to define relational semantics for the classical monotone modal logic introduced in the previous part of the current section.

Remark 5.4. The semantics on classical neighborhood frame $\langle W, N\rangle$ introduced in this section coincides with the semantics on $\langle W, =, N\rangle$ defined in Section 3.

To see this, we have to identify formulas of $\mathbf{L}([\,], [\,\rangle, \langle\,], \langle\,\rangle)$ and of $\mathbf{L}(\Box, \Diamond)$, and regard a classical neighborhood model as an intuitionistic neighborhood model.

First, we translate formulas from $\mathbf{L}([\,], [\,\rangle, \langle\,], \langle\,\rangle)$ to $\mathbf{L}(\Box, \Diamond)$ being based on the observation in Remark 5.3. We translate $[\,]$ and $[\,\rangle$ into $\Box$ and $\Diamond$, respectively. Other two modalities $\langle\,]$ and $\langle\,\rangle$ are duals of $[\,\rangle$ and $[\,]$, so it can be translated into $\neg\Diamond\neg$ and $\neg\Box\neg$, respectively.

As for models, we can regard $\langle W, N\rangle$ as $\langle W, =, N\rangle$, where $=$ is the equality on the set W. Since any valuation is admissible in $\langle W, =, N\rangle$, valuations on $\langle W, N\rangle$ and valuations on $\langle W, =, N\rangle$ are the same. It is easy to see that

$$\langle W, N\rangle, V, x \Vdash A \iff \langle W, =, N\rangle, V, x \Vdash_{\mathrm{n}} A',$$

where $A \in \mathbf{L}([\,], [\,\rangle, \langle\,], \langle\,\rangle)$, and $A' \in \mathbf{L}(\Box, \Diamond)$ is its correspondent (as described above).

Based on this observation, in what follows, we consider $[\,]$ and $[\,\rangle$ as the only primitive modalities, and denote them by $\Box$ and $\Diamond$.

Let $\langle W, N\rangle$ be a classical neighborhood frame, and regard this as $\langle W, =, N\rangle$. Then we can apply the transformation (3) to obtain $\langle W^*, \leq^*, \ni^*\rangle$. Here $\leq^*$ is given by

$$(x, X) \leq^* (y, Y) \iff x = y,$$

so $\leq^*$ becomes an equivalence relation.

The definition of the interpretation of formulas is the same as in Section 2. Since $\leq^*$ is an equivalence relation, and an interpretation is hereditary, the semantics is defined modulo this equivalence.

By abstracting these observations, we obtain the following definition.

Definition 5.5. An intuitionistic relational frame $\langle W, \simeq, R\rangle$ is said to be *degenerate* if $\simeq$ is an equivalence relation on W.

Definition 5.6. An intuitionistic relational model $\langle \mathcal{R}, V\rangle$ is said to be *degenerate* if $\mathcal{R}$ is degenerate.

Since degenerate intuitionistic relational frame is just a special case of intuitionistic relational frame, we can interpret modal formulas in this frame in the same way as the intuitionistic case.

Sound and complete axiomatization is obtained by adding the principle of excluded middle to the intuitionistic case.

Theorem 5.7. *A formula* $A \in \mathbf{L}(\Box, \Diamond)$ *is a theorem in* IM+PEM+$\mathrm{N}_\Diamond$ *if and only if* A *is true in all degenerate intuitionistic relational models.*

5.3 An Embedding into S5 $\otimes$ K

A relational semantics for monotone modal logic given above suggests that the monotone modal logic can be embedded into a normal bimodal logic. Since a degenerate intuitionistic relational frame has two binary relations corresponding to S5 and K, it is natural to think of a translation from the monotone logic to a bimodal logic obtained by combining S5 and K.

To formalize this idea, we first determine the translation image, which we call S5 $\otimes$ K.

A frame for $S5 \otimes K$ is just a degenerate intuitionistic relational frame. However, a notion of valuation is not the same.

Definition 5.8. An $(S5 \otimes K)$-model is a pair $\langle \mathcal{R}, V \rangle$ of a degenerate intuitionistic relational frame and a (not necessarily admissible) valuation V.

Given an $(S5 \otimes K)$-model, we can define a satisfaction relation, denoted by $\Vdash_b$. The boolean connectives are interpreted in the usual way, and modalities are interpreted as follows:

$$\mathcal{R}, V, x \Vdash_b \Box_1 A \iff \forall y.(x \simeq y \implies \mathcal{R}, V, y \Vdash_b A);$$
$$\mathcal{R}, V, x \Vdash_b \Box_2 A \iff \forall y.(x \mathrel{R} y \implies \mathcal{R}, V, y \Vdash_b A).$$

Proposition 5.9. *The logic* $S5 \otimes K$ *is axiomatized by the following axioms and inference rules:*

- *modus ponens;*
- *necessitation for both* $\Box_1$ *and* $\Box_2$*;*
- *classical tautology instances;*
- *axioms K, T, and 5 for* $\Box_1$*;*
- *axiom K for* $\Box_2$.

Proof. By the canonical model construction [3, Section 4.2]. □

Now we can define the translation from $\mathbf{L}(\Box, \Diamond)$ to $\mathbf{L}(\Box_1, \Box_2)$.

Definition 5.10. For each $A \in \mathbf{L}(\Box, \Diamond)$, define $|A|$ as follows:

$$|p| = \Box_1 p; \quad |\bot| = \Box_1 \bot;$$
$$|A * B| = |A| * |B|; \quad (* \text{ is either } \wedge, \vee, \text{ or } \to)$$
$$|\Box A| = \Box_1 \Box_2 |A|; \quad |\Diamond A| = \Box_1 \Diamond_2 |A|.$$

Definition 5.11. Let V be a valuation on a degenerate intuitionistic relational frame. Then its *admissible variant*, denoted by V°, is defined by

$$V^\circ(p) = \{x \mid \forall y \simeq x.\, y \in V(p)\}.$$

Lemma 5.12. *For all formulas $A \in \mathbf{L}(\Box, \Diamond)$ and a valuation V,*

$$\mathcal{R}, V, x \Vdash_{\mathrm{b}} |A| \iff \mathcal{R}, V^\circ, x \Vdash_{\mathrm{b}} |A| \iff \mathcal{R}, V^\circ, x \Vdash_{\mathrm{r}} A.$$

Proof. By induction on A. □

Theorem 5.13. *Let $A \in \mathbf{L}(\Box, \Diamond)$. Then, A is true in all degenerate intuitionistic relational models if and only if $|A|$ is true in all* (S5 ⊗ K)*-models.*

Proof.

Let us write $\mathcal{R}, V \Vdash_{\mathrm{r}} A$ if $\mathcal{R}, V, x \Vdash_{\mathrm{r}} A$ for all x, and similarly for $\mathcal{R}, V \Vdash_{\mathrm{b}} A$. Then the condition

A is true in all degenerate intuitionistic relational models

is rephrased as

$\mathcal{R}, V \Vdash_{\mathrm{r}} A$ for any $\mathcal{R}$ and admissible V,

where $\mathcal{R}$ ranges over all degenerate intuitionistic relational frames, and V ranges over all $\mathcal{R}$-valuations. This is equivalent to

$\mathcal{R}, V^\circ \Vdash_{\mathrm{r}} A$ for any $\mathcal{R}$ and (not necessarily admissible) V,

since V° is admissible for every V, and every admissible valuation V is an admissible variant of some valuation (because $V = V^\circ$ if V is admissible). By using the previous lemma, we can rewrite this condition into

$\mathcal{R}, V \Vdash_{\mathrm{b}} |A|$ for any $\mathcal{R}$ and (not necessarily admissible) V,

and this is the same as

$|A|$ is true in all $(\mathrm{S5} \otimes \mathrm{K})$-models.

□

6. Concluding Remarks

6.1 Summary

We have investigated semantic aspects of intuitionistic modal logics without distributivity law $\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$. We have defined neighborhood semantics, and proved that the existing relational semantics can be represented in terms of neighborhood semantics, as well as the converse under a slight restriction. This shows a close relationship between these two semantics.

By using this result, we have also considered the relationship between classical monotone modal logic and normal bimodal logic. We proved that the classical monotone modal logic with $\mathrm{N}_{\Diamond}$ has relational representation of its neighborhood semantics, and embeddable into $\mathrm{S5} \otimes \mathrm{K}$.

The results obtained from these investigations bring us a new insight in intuitionistic modal logic and classical non-normal modal logic. In particular, it turned out that (some of) the existing intuitionistic modal logic can actually be captured as an intuitionistic version of non-normal modal logic in a natural way.

6.2 Non-Normal Modalities and Multimodal Logics

Translation from non-normal modal logics to normal multimodal logics has already been studied before. Gasquet and Herzig showed that non-normal (not necessarily monotone) modal logic can be translated into normal modal logic with three modalities [7]. Kracht and Wolter proved that monotone modal logic can be "simulated" by normal bimodal logic (actually, they also proved that a normal monomodal logic can simulate monotone modal logics) [10].

The basic idea behind their work is different from ours. Our translation from monotone to bimodal logic is based on the idea of considering the set

$$W^* = \{(x, X) \mid x \in W, X \in N(x)\},$$

which consists of all pairs of possible worlds and their neighborhoods. On the other hand, both of the previous approaches consider the set

$$W \cup \bigcup_{x \in W} N(x),$$

which consists of all possible worlds and all subsets of W that are neighborhoods of some worlds.[2]

This causes the difference in source and target logics of translations. In our translation, both of the source and target are stronger logics than the previous work. We assume the axiom $\mathrm{N}_\Diamond$ in the source logic, and considered $\mathrm{S5} \otimes \mathrm{K}$, a combination of S5 and K, as a target. In the previous work, they did not assume an extra axiom like $\mathrm{N}_\Diamond$, and the target is a combination of two (in the case of Kracht and Wolter) or three (in the case of Gasquet and Herzig) copies of K.

6.3 Relationship with Gödel Translation

Our translation from monotone modal logic to $\mathrm{S5} \otimes \mathrm{K}$ can be considered as a variant of Gödel translation. Wolter and Zakharyaschev [16] investigated an embedding from intuitionistic modal logic into classical normal bimodal logic. They defined a Gödel-style translation, denoted by t, from an intuitionistic modal logic (with $\Box$ as the only primitive modality) into $\mathrm{S4}\otimes\mathrm{K}$. Our translation $|\cdot|$ can be seen as a variant of theirs.

At first sight, there is a difference between these two translations in the case of implication. Wolter and Zakharyaschev's t is defined as

$$t(A \to B) = \Box_1(t(A) \to t(B)),$$

which is the same as the usual Gödel translation, while our version $|\cdot|$ is given by

$$|A \to B| = |A| \to |B|\,.$$

However, when $\Box_1$ is an S5 modality, this makes no difference; we can prove that $|A| \to |B|$ and $\Box_1(|A| \to |B|)$ are equivalent in $\mathrm{S5} \otimes \mathrm{K}$.

[2] Actually, Kracht and Wolter used more sophisticated technique, but the basic idea is as described here.

6.4 "Internal Observer" Interpretation

One possible interpretation that rejects the distributivity is to consider observers inside possible worlds, which we call "internal observer." Let us assume that an observer o_x is assigned to each possible world x. We will consider a formula A true at a world x if the observer o_x is able to verify that A is true.

$\Diamond(A \vee B)$ at x means that an observer o_x knows $A \vee B$ is true somewhere, say y. Note that this does not necessarily mean that o_x can determine the disjunct that becomes true at y. It is o_y who can know which of A and B is actually true at y. This means that, in view of o_x, neither $\Diamond A$ nor $\Diamond B$ cannot be verified to be true. Therefore $\Diamond(A \vee B)$ does not necessarily imply $\Diamond A \vee \Diamond B$, and this is why the internal observer interpretation rejects distributivity.

In the usual Kripke semantics, unlike this interpretation, we implicitly assume a viewpoint of an observer outside the Kripke frame. So we can say that it takes an external observer's viewpoint. The argument above would not be true if we take this point of view, and the distributivity cannot be rejected (indeed, the usual Kripke semantics admits distributivity).

Similar viewpoint can be found in Aucher's work on internal (and imperfect external) epistemic logic [2]. He investigated an epistemic logic based on the view from inside the situation, rather than the usual view from outside the situation. To model such a circumstance, Aucher considered disjoint sum of several Kripke models.

6.5 Neighborhoods as Ambiguity

The interpretation discussed above is partially expressed in neighborhood semantics. An internal observer o_x does not have complete information about other worlds, so o_x has several possibilities in mind about the actual situation of other worlds. Each of these possible situations is represented as a neighborhood. For example, in the situation above (o_x can verify $\Diamond(A \vee B)$ but not $\Diamond A \vee \Diamond B$), o_x would think of two possibilities about the sets of accessible worlds (which we call X and X'). In X a world that makes A true can be found, and X' contain a world that makes B true. This uncertainty can be expressed in terms of neighborhoods, that is, each neighborhood of x represents a possible set of accessible worlds from x in view of o_x. Incidentally, the decreasing condition on neighborhood function is understood as a natural assumption that the amount of uncertainty would decrease if the amount of knowledge increases.

Another way to express this uncertainty is to consider a pair (x, X), where x is a possible world, and X is a candidate of the set of accessible worlds from x. This construction is precisely what we did to transform a neighborhood model into a relational model. So the notion of possible worlds in relational semantics actually carries two pieces of information, the current state of knowledge and the set of accessible worlds.

A similar idea can be found in Hilken's work [8], which investigates Stone duality for modal frames (a complete Heyting algebras equipped with modal operators). In his theory, the notion of points in modal frames contains two components; a completely prime filter p and an element a of the frame such that $\Diamond a \notin p$.[3] The first component p is the same as the notion of point appearing in the duality theory between frames and spaces. The second component a carries an extra piece of information; intuitively, this is (the interior of) the set of points that are *not* accessible from the point (p, a) represents.

As we have seen above, a kind of uncertainty plays an important role in the semantics of intuitionistic modal logic we have considered in this paper (and the existing literature). This is the source of the difficulty when we try to capture intuitionistic modal logics in the framework of Kripke semantics, which is originally a framework for normal modal logics. What we have presented in this paper is that intuitionistic modal logics can be treated as non-normal modal logics rather than normal ones. We believe this finding advances our understanding of intuitionistic modal logics.

A. Proof of Completeness

Soundness can be checked by the standard induction, so we will prove the completeness only. The basic strategy of the proof is the same as the one found in Section 4 of [3], except that we consider neighborhood frames instead of relational frames.

[3] Precisely speaking, Hilken calls such a pair "pre-point," and defines the set of points as a certain subset of the set of all pre-points.

Lemma A.1 (deduction theorem)**.** *Let Λ be an $\mathbf{L}(\Box, \Diamond)$-logic and Γ a Λ-theory. Then, $A \to B \in \Gamma$ if and only if all Λ-theories Δ containing Γ and A contain B.*

Proof. To prove "if" part, let $\Delta = \{C \mid A \to C \in \Gamma\}$. Then Δ is a theory containing Γ and A. Then, from assumption, Δ contains B, and this means $A \to B \in \Gamma$ by definition of Δ.

For the "only if" part, use the fact that any theory is closed under modus ponens. $\Box$

Definition A.2. Let Λ be an $\mathbf{L}(\Box, \Diamond)$-logic and Γ a Λ-theory. Γ is said to be *prime* if it satisfies the following conditions:

1. if $A \vee B \in \Gamma$, then either $A \in \Gamma$ or $B \in \Gamma$;
2. $\bot \notin \Gamma$.

Lemma A.3 (extension lemma)**.** *Let Λ be an $\mathbf{L}(\Box, \Diamond)$-logic and Γ a Λ-theory not containing A. Then, there exists a prime Λ-theory Δ such that $A \notin \Delta$ and $\Gamma \subseteq \Delta$.*

Proof. A maximal element of $\{\Delta \mid \Gamma \subseteq \Delta, A \notin \Delta\}$ has the required property. $\Box$

Definition A.4. Let Λ be an $\mathbf{L}(\Box, \Diamond)$-logic and Γ a Λ-theory. Then we define $\Box^{-1}\Gamma := \{A \mid \Box A \in \Gamma\}$.

Lemma A.5. *Let Λ be an $\mathbf{L}(\Box, \Diamond)$-logic containing axiom K. If Γ is a Λ-theory, then so is $\Box^{-1}\Gamma$.*

Proof. If $A \to B \in \Box^{-1}\Gamma$ and $A \in \Box^{-1}\Gamma$, then $\Box(A \to B) \in \Gamma$ and $\Box A \in \Gamma$. Since Λ contains K, it follows that $\Box B \in \Gamma$, hence $B \in \Box^{-1}\Gamma$. So $\Box^{-1}\Gamma$ is closed under modus ponens. $\Box$

Below, we will construct a canonical model for a logic Λ and prove standard properties. In what follows, we fix an arbitrary $\mathbf{L}(\Box, \Diamond)$-logic Λ containing $\mathrm{IM} + \mathrm{N}_{\Diamond\Box}$.

Definition A.6 (canonical neighborhood frame)**.** We define $\mathcal{N}^{\Lambda}$ to be a tuple $\langle W^{\Lambda}, \leq^{\Lambda}, N^{\Lambda} \rangle$, where

- W^{Λ} is the set of all prime Λ-theories,

- $\leq^\Lambda$ is the inclusion relation between sets, and
- N^Λ is defined by

$$N^\Lambda(\Gamma) = \left\{ n(\Gamma', A) \,\middle|\, \Gamma \subseteq \Gamma' \in W^\Lambda, \Diamond A \notin \Gamma' \right\},$$
$$n(\Gamma, A) = \left\{ \Delta \in W^\Lambda \,\middle|\, \Box^{-1}\Gamma \subseteq \Delta \text{ and } A \notin \Delta \right\}$$

for each $\Gamma \in W^\Lambda$.

Definition A.7 (canonical valuation)**.** A valuation V^Λ on $\mathcal{N}^\Lambda$ is defined by

$$V^\Lambda(p) = \{\Gamma \mid p \in \Gamma\}.$$

Definition A.8 (canonical model)**.** Clearly V^Λ is admissible, so $\mathcal{M}^\Lambda = \langle \mathcal{N}^\Lambda, V^\Lambda \rangle$ is an intuitionistic neighborhood model. We call this model a *canonical model for* Λ.

As usual, the following holds.

Theorem A.9. $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} A$ *if and only if* $A \in \Gamma$.

Proof. We proceed by induction on A. The cases when A is an atomic formula, $\bot$, conjunction, and disjunction are trivial.

$\underline{A \to B \in \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} A \to B}$: For all $\Delta \geq^\Lambda \Gamma$, if $\mathcal{N}^\Lambda, V^\Lambda, \Delta \Vdash_{\mathrm{n}} A$, then $A \in \Delta$ by induction hypothesis, so $A, A \to B \in \Delta$. Since Δ is closed under modus ponens, we obtain $B \in \Delta$. This means that $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} B$ by induction hypothesis.

$\underline{A \to B \notin \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \nVdash_{\mathrm{n}} A \to B}$: Suppose $A \to B \notin \Gamma$. Then, by deduction theorem and extension lemma, there exists $\Delta \in W^\Lambda$ such that $A \in \Delta$, $\Gamma \subseteq \Delta$, and $B \notin \Delta$. So $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \nVdash_{\mathrm{n}} A \to B$.

$\underline{\Box A \in \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} \Box A}$: If $\Delta \in n(\Gamma', B) \in N^\Lambda(\Gamma)$ for some Γ' and B, then $\Box^{-1}\Gamma \subseteq \Box^{-1}\Gamma' \subseteq \Delta$. Since $\Box A \in \Gamma$, we have $A \in \Box^{-1}\Gamma \subseteq \Delta$. Therefore $\mathcal{N}^\Lambda, V^\Lambda, \Delta \Vdash_{\mathrm{n}} A$. Since this holds for all Γ', B and Δ, it follows that $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} \Box A$.

$\underline{\Box A \notin \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \nVdash_{\mathrm{n}} \Box A}$: First, note that $\Box A \notin \Gamma$ means $\Diamond\bot \notin \Gamma$, since

$$\Diamond\bot \to \Box\bot, \Box\bot \to \Box A \in \Gamma$$

from $\mathrm{N}_{\Diamond\Box}$ and the monotonicity of $\Box$. This means that $n(\Gamma, \bot) \in N^{\Lambda}(\Gamma)$. So it suffices to show that $n(\Gamma, \bot)$ contains some Δ such that $A \notin \Delta$. Such Δ can be obtained as follows. Since $\Box A \notin \Gamma$, we have $A \notin \Box^{-1}\Gamma$. By using extension lemma we can obtain a prime Λ-theory $\Delta \supseteq \Box^{-1}\Gamma$ with $A \notin \Delta$. For such Δ, it holds that $\Delta \in n(\Gamma, \bot)$.

$\underline{\Diamond A \in \Gamma \implies \mathcal{N}^{\Lambda}, V^{\Lambda}, \Gamma \Vdash_{\mathrm{n}} \Diamond A}$: Take an arbitrary $n(\Gamma', B) \in N^{\Lambda}(\Gamma)$. Then we have $\Diamond B \notin \Gamma'$. Let Θ be the least theory containing $\Box^{-1}\Gamma'$ and A.

We first show that $B \notin \Theta$. If $B \in \Theta$, then we would have $A \to B \in \Box^{-1}\Gamma'$ from deduction theorem. This means $\Box(A \to B) \in \Gamma'$, hence $\Diamond A \to \Diamond B \in \Gamma'$ since

$$\Box(A \to B) \to \Diamond A \to \Diamond B \in \Gamma'.$$

However, $\Diamond A \in \Gamma \subseteq \Gamma'$ from assumption, so it follows that $\Diamond B \in \Gamma'$, a contradiction.

Now we have $\Box^{-1}\Gamma' \subseteq \Theta$, $A \in \Theta$, and $B \notin \Theta$. By extension lemma, there exists Δ satisfying the same conditions. For such Δ, we have $\Delta \in n(\Gamma', B)$, and $A \in \Delta$, and hence $\mathcal{N}^{\Lambda}, V^{\Lambda}, \Delta \Vdash_{\mathrm{n}} A$.

So we have proved that for all neighborhood $n(\Gamma', B)$ of Γ there exists $\Delta \in n(\Gamma', B)$ such that $\mathcal{N}^{\Lambda}, V^{\Lambda}, \Delta \Vdash_{\mathrm{n}} A$. This means that $\mathcal{N}^{\Lambda}, V^{\Lambda}, \Gamma \Vdash_{\mathrm{n}} \Diamond A$.

$\underline{\Diamond A \notin \Gamma \implies \mathcal{N}^{\Lambda}, V^{\Lambda}, \Gamma \nVdash_{\mathrm{n}} \Diamond A}$: Assume $\Diamond A \notin \Gamma$, and let $X = n(\Gamma, A)$. Clearly X is a neighborhood of Γ, and any of its element Δ does not contain A. This means that $\mathcal{N}^{\Lambda}, V^{\Lambda}, \Delta \nVdash_{\mathrm{n}} A$ for all $\Delta \in X$. Therefore $\mathcal{N}^{\Lambda}, V^{\Lambda}, \Gamma \nVdash_{\mathrm{n}} \Diamond A$.

□

The following is an easy consequence of this theorem.

Lemma A.10. *Let $\mathcal{K}$ be a class of intuitionistic neighborhood models. If $\mathcal{M}^{\Lambda} \in \mathcal{K}$, then Λ is complete with respect to $\mathcal{K}$, that is, any formula true in $\mathcal{K}$ is a theorem of Λ.*

By using this lemma, the completeness parts of Theorem 3.6 and Theorem 3.9 can be reduced to the following lemma, which is easily verified.

Lemma A.11. *1. The canonical model of* $\mathrm{IM}+\mathrm{N}_{\Diamond\Box}$ *is an intuitionistic neighborhood model.*

2. The canonical model of $\mathrm{IM}+\mathrm{N}_{\Diamond}$ *is a normal intuitionistic neighborhood model.*

Proof. The first claim is immediate from the definition of N^{Λ}. For the second part, it suffices to show that $N^{\mathrm{IM}+\mathrm{N}_{\Diamond}}(\Gamma) \neq \emptyset$ for each Γ. Actually, $n(\Gamma, \bot)$ is always a neighborhood of Γ. This follows from the presence of $\mathrm{N}_{\Diamond}$: any prime Γ does not contain $\Diamond\bot$ since $\neg\Diamond\bot \in \Gamma$ and $\bot \notin \Gamma$. □

Next, we will prove Theorem 5.7. Here, we identify a classical neighborhood frame $\langle W, N\rangle$ and an intuitionistic neighborhood frame $\langle W, =, N\rangle$, and similarly for a classical neighborhood model. The following is also immediate from Lemma A.10.

Lemma A.12. *A formula* $A \in \mathbf{L}(\Box, \Diamond)$ *is a theorem of* $\mathrm{IM}+\mathrm{PEM}+\mathrm{N}_{\Diamond}$ *if and only if it is true in all normal classical neighborhood models.*

Proof. It is straightforward to verify that the canonical model of $\mathrm{IM} + \mathrm{PEM} + \mathrm{N}_{\Diamond}$ is a normal classical neighborhood model. Therefore, completeness follows from Lemma A.10. Soundness is proved in the usual way. □

So it suffices to prove that degenerate intuitionistic relational models and normal classical neighborhood models determine the same logic. Basically this is done by mutual translations between models presented in Section 4, but there is a subtle problem. If $\langle \mathcal{N}, V\rangle$ is a normal classical neighborhood model, then its translation is a degenerate intuitionistic relational model, so this direction is straightforward. Consider the other direction. If we have a degenerate intuitionistic relational model $\langle \mathcal{R}, V\rangle$, by translation we obtain a model $\langle \mathcal{N}_{\mathcal{R}}, V\rangle$, which is not necessarily classical. A neighborhood frame is classical when its $\leq$-part is the equality $=$, but in this case this is not the case (it is only an equivalence relation).

Actually, this is not a big problem. We can fix this by considering quotient of $\mathcal{N}_{\mathcal{R}}$, which is indeed a classical neighborhood frame. In general, we can prove the following:

Proposition A.13. *Let* $\mathcal{N} = \langle W, \leq, N\rangle$ *be an intuitionistic neighborhood frame, and* V *an admissible* $\mathcal{N}$*-valuation. Define its quotient* $|\mathcal{N}| = \langle |W|, |\leq|, |N|\rangle$ *and* $|V|$ *as follows.*

- *$|W| = W/{\sim}$, where $x \sim y$ if and only if $x \leq y$ and $y \leq x$.*
- *$[x] \; |{\leq}| \; [y]$ if and only if $x \leq y$, where $[z]$ denotes the equivalence class of z. This does not depend on the choice of x and y.*
- *$|N|([x]) = \{X/{\sim} \mid X \in N(x)\}$, where $X/{\sim}$ is the image of X under the canonical projection $W \to |W|$. Since N is decreasing, $x \sim y$ implies $N(x) = N(y)$, so $|N|$ is well-defined.*
- *$|V|(p) = V(p)/{\sim}$.*

Then, for any $A \in \mathbf{L}(\Box, \Diamond)$, we have

$$\mathcal{N}, V, x \Vdash_{\mathrm{n}} A \iff |\mathcal{N}|, |V|, [x] \Vdash_{\mathrm{n}} A.$$

Proof. By induction on A. □

For any intuitionistic neighborhood frame $\mathcal{N} = \langle W, \leq, N \rangle$, preorder structure $|{\leq}|$ of its quotient $|\mathcal{N}|$ is clearly an order, that is, it is anti-symmetric. In particular, when $\leq$ is an equivalence relation, $|{\leq}|$ is the equality on the quotient $|W|$. Applying this construction to $\mathcal{N}_{\mathcal{R}}$, we can check that each degenerate intuitionistic relational model has an equivalent normal classical neighborhood model $|\mathcal{N}|_{\mathcal{R}}$. This completes the proof of Theorem 5.7.

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