

Proposition 21. *LLPO and 1-König are equivalent.*

We next formulate the *Weak Lesser Limited Principle of Omniscience* (WLLPO):

WLLPO $\neg(\neg A_0 \wedge \neg A_1) \Rightarrow \neg\neg A_0 \vee \neg\neg A_1$ for all simply existential assertions A_0, A_1 .

For proofs that LLPO implies WLLPO using real numbers and binary sequences see [20, Theorem 4.1, Theorem 4.2] and [14, Proposition 1.2], respectively. While WLLPO occurs as LLPE in the former reference, in the latter LLPO and WLLPO are called SEP and MP^\vee , respectively. Our choice of the name for WLLPO is motivated by the fact that WLLPO is to LLPO just as WLPO is to LPO. More precisely, WLLPO is tantamount to

$$\neg(B_0 \wedge B_1) \Rightarrow \neg B_0 \vee \neg B_1 \text{ for all simply universal assertions } B_0, B_1,$$

the counterpart of LLPO for simply universal assertions rather than simply existential ones. Note also that WLLPO is equivalent to the statement that, for every tree T ,

$$\neg(T_0 \text{ infinite} \wedge T_1 \text{ infinite}) \Rightarrow T_0 \text{ not infinite} \vee T_1 \text{ not infinite},$$

just as LLPO was characterised in Proposition 19.

Furthermore, WLLPO follows from MP, which is called LPE in [20]. More precisely, MP is equivalent to WLLPO in conjunction with the weak Markov principle WMP [14, Proposition 1.1]—or, in other terms, LPE is equivalent to LLPE plus WLPE [20, Section 1]. Also, WMP is a consequence of a form of Church’s thesis for disjunctions, under which it thus is equivalent to MP [14, Proposition 2, Theorem 1].

Here is another approach to the implications from LLPO and from MP to WLLPO. We say that a decidable assertion $D(n)$ holds for at most one n if $m = n$ whenever $D(m)$ and $D(n)$. It is well-known that LLPO and MP can equivalently put as follows:

LLPO₀ *If a decidable assertion $D(n)$ holds for at most one n , then there is $i \in \{0, 1\}$ with $\forall k \neg D(2k + i)$.*

MP₀ For every decidable assertion $D(n)$ which holds for at most one n , if $\neg\forall n \neg D(n)$, then there is $i \in \{0, 1\}$ with $\exists k D(2k + i)$.

A similar characterisation of WLLPO was inspired by the proof of [14, Proposition 1] and by a remark in [19, Section 3]:⁴

WLLPO₀ For every decidable assertion $D(n)$ which holds for at most one n , if $\neg\forall n \neg D(n)$, then there is $i \in \{0, 1\}$ with $\forall k \neg D(2k + i)$.

In MP₀ and WLLPO₀ the index $i \in \{0, 1\}$ is uniquely determined by the required property.

Lemma 22. WLLPO and WLLPO₀ are equivalent.

Proof. For each $i \in \{0, 1\}$ set $\bar{i} = 1 - i$. Assume first WLLPO, and let $D(n)$ be a decidable assertion. For each $i \in \{0, 1\}$ set

$$A_i \equiv \exists k D(2k + i),$$

for which $\neg A_0 \wedge \neg A_1$ is equivalent to $\forall n \neg D(n)$. If $\neg\forall n \neg D(n)$, which is to say that $\neg(\neg A_0 \wedge \neg A_1)$, then $\neg\neg A_0 \vee \neg\neg A_1$ by WLLPO. If $D(n)$ holds for at most one n , then $\neg A_{\bar{i}}$ follows from A_i , and thus already from $\neg\neg A_i$.

Assume next WLLPO₀, and let A_i be a simply existential assertion of the form

$$A_i \equiv \exists k D_i(k)$$

where $D_i(k)$ is a decidable assertion for each $i \in \{0, 1\}$. Define decidable assertions $E(n)$ and $F(n)$ by

$$E(2k + i) \equiv D_i(k)$$

for every k and each $i \in \{0, 1\}$, and by

$$F(n) \equiv E(n) \wedge \forall m < n \neg E(m)$$

⁴This remark says that the following statements are equivalent:

$$\forall x \in \mathbb{R} (\neg(x \leq 0 \wedge x \geq 0) \Rightarrow \neg(x \leq 0) \vee \neg(x \geq 0)) ;$$

$$\forall x \in \mathbb{R} (\neg(x \leq 0 \wedge x \geq 0) \Rightarrow x \leq 0 \vee x \geq 0) .$$

Note in this context that $x \leq y$ is a simply universal statement for every pair $x, y \in \mathbb{R}$, for it is the negation of the simply existential statement $x > y$.

for every n . Note that $F(n)$ holds for at most one n , and that A_i is equivalent to $\exists k E(2k+i)$. In particular, $\neg A_0 \wedge \neg A_1$ is equivalent to $\forall n \neg E(n)$, and $\exists k F(2k+i)$ implies A_i . Moreover, $\exists n E(n)$ implies $\exists n F(n)$; whence $\forall n \neg F(n)$ implies $\forall n \neg E(n)$. Now if $\neg(\neg A_0 \wedge \neg A_1)$, then $\neg \forall n \neg F(n)$; whence if $\forall k \neg F(2k+i)$, then $\neg \forall k \neg F(2k+i)$, and thus $\neg \neg A_{\bar{i}}$ for each $i \in \{0, 1\}$. \square

With LLPO and MP as LLPO_0 and MP_0 , respectively, the following is evident:

Corollary 23. *Each of LLPO and MP implies WLLPO .*

Proposition 24. *Each of the following statements is equivalent to WLLPO :*

1. $\neg(\neg A_0 \wedge \neg A_1) \wedge \neg(A_0 \wedge A_1) \Rightarrow \neg \neg A_0 \vee \neg \neg A_1$ for all simply existential assertions A_0, A_1 ;
2. $\neg(\neg A_0 \wedge \neg A_1) \wedge \neg(A_0 \wedge A_1) \Rightarrow (\neg \neg A_0 \vee \neg \neg A_1) \wedge (\neg A_1 \vee \neg A_0)$ for all simply existential assertions A_0, A_1 ;
3. $\neg(\neg A_0 \wedge \neg A_1) \wedge \neg(A_0 \wedge A_1) \Rightarrow \neg A_1 \vee \neg A_0$ for all simply existential assertions A_0, A_1 .

Proof. The following are equivalent: $\neg(B \wedge C); B \Rightarrow \neg C; \neg \neg B \Rightarrow \neg C$. In particular, items 1, 2, and 3 are equivalent. It is obvious that WLLPO implies item 1, whereas WLLPO as characterised in Lemma 22 is readily deduced from item 3: apply 3 to $A_i \equiv \exists k D(2k+i)$ whenever $D(n)$ is a decidable assertion. \square

In view of its equivalent MP_0 , it is clear that MP is tantamount to

$$\neg(\neg A_0 \wedge \neg A_1) \Rightarrow A_0 \vee A_1 \text{ for all simply existential assertions } A_0, A_1.$$

In particular, MP is equivalent to the statement that, for every tree T ,

$$\neg(T_0 \text{ infinite} \wedge T_1 \text{ infinite}) \Rightarrow T_0 \text{ finite} \vee T_1 \text{ finite}.$$

Moreover, one can characterise MP just as WLLPO was treated in Proposition 24:

Proposition 25. *Each of the following statements is equivalent to MP :*

1. $\neg(\neg A_0 \wedge \neg A_1) \wedge \neg(A_0 \wedge A_1) \Rightarrow A_0 \vee A_1$ for all simply existential assertions A_0, A_1 ;
2. $\neg(\neg A_0 \wedge \neg A_1) \wedge \neg(A_0 \wedge A_1) \Rightarrow (A_0 \vee A_1) \wedge (\neg A_1 \vee \neg A_0)$ for all simply existential assertions A_0, A_1 .

6. The weak König lemma and dependent choice

In [13], WKL and LLPO were shown to be equivalent. While no other principle was used for deducing LLPO from WKL (see also our Corollary 20), dependent choice was taken for granted for proving the implication from LLPO to WKL. In [1, 2.2] it was pointed out that LLPO is equivalent to WKL with respect to a binary choice for simply universal formulas. Also using dependent choice, a deduction of WKL from 1-König—which is an equivalent of LLPO (see Proposition 21 above and the discussion preceding it)—was given in [4, Footnote 2].

Following [15, 16.5], we next decompose WKL into LLPO and a weak form of dependent choice. The following equivalence, with LLPO named Σ_1^0 -DML, was proved in [15, 16.5]:

$$\text{WKL} \Leftrightarrow \text{LLPO} + \Pi_1^0\text{-CC}^\vee. \quad (5)$$

The proof of (5) given in [15, 16.5] goes through a third equivalent of a somewhat topological character and requires some coding. This can be avoided, as follows, if one uses $\Pi_1^0\text{-DC}^\vee$ in place of $\Pi_1^0\text{-CC}^\vee$. The proofs of the next two results are unwindings of the proof of [31, Lemma IV.4.4].

Theorem 26. *WKL implies $\Pi_1^0\text{-DC}^\vee$.*

Proof. Assume WKL. To prove $\Pi_1^0\text{-DC}^\vee$ let $A_0(u)$ and $A_1(u)$ be simply universal for every u : that is, A_i is of the form $\forall k D_i(u, k)$ where $D_i(u, k)$ is a decidable assertion for each $i \in \{0, 1\}$. Set

$$T = \{u : \forall n < |u| \forall k < |u| D_{u(n)}(\bar{u}n, k)\},$$

which clearly is a tree. For each α we have

$$\forall m \forall n < m \forall k < m D_{\alpha(n)}(\bar{\alpha}n, k) \Leftrightarrow \forall n \forall k D_{\alpha(n)}(\bar{\alpha}n, k)$$

and thus

$$\forall m (\bar{\alpha}m \in T) \Leftrightarrow \forall n A_{\alpha(n)}(\bar{\alpha}n).$$

In other words, an infinite path in T is nothing but an infinite sequence α as in the conclusion of DC^\vee . By WKL it therefore suffices to show that T is infinite whenever the hypothesis of DC^\vee holds. To this end, consider

$$S = \{u : \forall n < |u| A_{u(n)}(\bar{u}n)\} = \{u : \forall n < |u| \forall k D_{u(n)}(\bar{u}n, k)\},$$

and observe that $S \subseteq T$. Now if $A_i(u)$, then $ui \in S$; whence if $\forall u (A_0(u) \vee A_1(u))$, then

$$\forall u (u \in S \Rightarrow u0 \in S \vee u1 \in S).$$

By induction, for every m there is u with $|u| = m$ such that $u \in S$ and thus $u \in T$. \square

Theorem 27. *WKL follows from $\text{LLPO} + \Pi_1^0\text{-DC}^\vee$.*

Proof. Assume LLPO and $\Pi_1^0\text{-DC}^\vee$. To deduce WKL, let T be a tree and set

$$D(m, u) \equiv \exists v (|v| = m \wedge v \in T_u).$$

This is a decidable assertion and satisfies

$$\forall m D(m, u) \Leftrightarrow T_u \text{ infinite}$$

for every u . For each $i \in \{0, 1\}$ we set $\bar{i} = 1 - i$ and define

$$A_i(u) \equiv \exists m (\neg D(m, ui) \wedge D(m, u\bar{i})),$$

which is simply existential for every u .

We next show that $\forall u \neg(A_0(u) \wedge A_1(u))$. Assume that $A_0(u) \wedge A_1(u)$, which is to say that there are m_0 and m_1 with $\neg D(m_0, u0)$, $D(m_0, u1)$, $\neg D(m_1, u1)$, and $D(m_1, u0)$. Suppose that $m_0 \geq m_1$. By $D(m_0, u1)$ there is $v \in T_{u1}$ with $|v| = m_0$, so that for $w = \bar{v}m_1$ we have $w \in T_{u1}$ and $|w| = m_1$ in contradiction to $\neg D(m_1, u1)$. The case $m_0 \leq m_1$ can be treated in the same way, using first $D(m_1, u0)$ and then $\neg D(m_0, u0)$.

By LLPO as characterised in Lemma 16 we thus have $\forall u (\neg A_0(u) \vee \neg A_1(u))$; whence by $\Pi_1^0\text{-DC}^\vee$ there is α such that $\forall n \neg A_{\alpha(n)}(\bar{\alpha}n)$. In view of Lemma 18, it now suffices to show that if T is infinite, then $T_{\bar{\alpha}n}$ is infinite

for this α and every n . We proceed by induction on n , using that $T_{\bar{\alpha}n}$ is infinite precisely when $\forall m D(m, \bar{\alpha}n)$.

The case $n = 0$ amounts to T being infinite.

To deduce $\forall m D(m, \bar{\alpha}(n+1))$ from $\forall m D(m, \bar{\alpha}n)$, suppose the latter, and fix an arbitrary m . By the very definition of $D(m+1, \bar{\alpha}n)$ we have $D(m, (\bar{\alpha}n)0)$ or $D(m, (\bar{\alpha}n)1)$. If both alternatives hold, then clearly $D(m, \bar{\alpha}(n+1))$. If, however, $\neg D(m, (\bar{\alpha}n)i)$ for some $i \in \{0, 1\}$, then $D(m, (\bar{\alpha}n)\bar{i})$ and $A_i(\bar{\alpha}n)$; whence $\alpha(n) = \bar{i}$ (by the choice of α) and again $D(m, \bar{\alpha}(n+1))$. \square

Corollary 28. *WKL is equivalent to $\text{LLPO} + \Pi_1^0\text{-DC}^\vee$.*

A tree S is a *spread* if every element of S has an immediate successor in S : that is,

$$\forall u (u \in S \Rightarrow u0 \in S \vee u1 \in S) \quad (6)$$

or, equivalently,

$$\forall u ((u \in S \Rightarrow u0 \in S) \vee (u \in S \Rightarrow u1 \in S)) \quad (7)$$

(recall that every tree is assumed to be detachable). By induction every spread is an infinite tree; whence WKL implies a principle that we therefore call the *Weak Spread Lemma* (WSL):

WSL *Every spread has an infinite path.*

For Kleene's time-honoured discovery [17] of an infinite tree without infinite path in recursive mathematics, one cannot expect to prove WKL with constructive means. Its consequence WSL, however, is weak enough to allow for a constructive proof:

Proposition 29. *WSL is provable.*

Proof. Since $\Delta_0\text{-DC}^\vee$ is provable (Proposition 3), we only need to show that it implies WSL. To this end, let S be a spread. For each $i \in \{0, 1\}$ set

$$A_i(u) \equiv (u \in S \Rightarrow ui \in S),$$

which is a decidable assertion. By (7) we have $\forall u (A_0(u) \vee A_1(u))$; whence by $\Delta_0\text{-DC}^\vee$ there is α with $\forall n A_{\alpha(n)}(\bar{\alpha}n)$. Induction on n proves that

$\forall n (\bar{\alpha}n \in S)$ for this α . (The case $n = 0$ is $() \in S$; if $\bar{\alpha}n \in S$, then $\bar{\alpha}(n+1) \in S$ for $\bar{\alpha}(n+1) = (\bar{\alpha}n)\alpha(n)$ and $A_{\alpha(n)}(\bar{\alpha}n)$.) \square

In this proof we have inferred WSL from $\Delta_0\text{-DC}^\vee$. Conversely, $\Delta_0\text{-DC}^\vee$ can be deduced from WSL as follows. Let $A_0(u)$ and $A_1(u)$ be decidable assertions. Set

$$S = \{u : \forall k < |u| A_{u(k)}(\bar{u}k)\},$$

which clearly is a tree. If $\forall u (A_0(u) \vee A_1(u))$, then (6) holds for this S , because $u \in S$ together with $A_i(u)$ implies $ui \in S$ for $i \in \{0, 1\}$. By WSL there is α with $\forall n (\bar{\alpha}n \in S)$: that is, $A_{\alpha(k)}(\bar{\alpha}k)$ for every $k < n$ and all n , or simply $\forall n A_{\alpha(n)}(\bar{\alpha}n)$.

7. Omniscience principles put in a uniform way

As a complement we rephrase in a uniform way all the omniscience principles but Markov's that have occurred in this paper. To this end we need to fix the *Law of the Excluded Middle* ($\Gamma\text{-LEM}$) and *De Morgan's Law* ($\Gamma\text{-DML}$) as restricted to any assertion class Γ :

$\Gamma\text{-LEM}$ $C \vee \neg C$ for all $C \in \Gamma$;

$\Gamma\text{-DML}$ $\neg(C \wedge D) \Rightarrow \neg C \vee \neg D$ for all $C, D \in \Gamma$.

For arbitrary assertion classes Γ, Δ we consider the following principle:

$P(\Gamma, \Delta)$ $(C \Rightarrow D) \Rightarrow \neg C \vee D$ for all $C \in \Gamma$ and $D \in \Delta$.

We further write Φ for the class of all assertions. The proof of the next lemma is left to the reader as an exercise in intuitionistic propositional logic.

Lemma 30. *Let Γ be a class of assertions.*

1. *The following are equivalent: $\Gamma\text{-LEM}$; $P(\Gamma, \Phi)$; $P(\Phi, \Gamma)$; $P(\Gamma, \Gamma)$.*
2. *$\Gamma\text{-DML}$ is equivalent to $P(\Gamma, \Delta)$ with $\Delta = \{\neg C : C \in \Gamma\}$.*

Note that LPO, WLPO, LLPO, and WLLPO are nothing but Σ_1^0 -LEM, Π_1^0 -LEM, Σ_1^0 -DML and Π_1^0 -DML, respectively. We set

$$\Xi_1^0 = \{\neg B : B \in \Pi_1^0\} = \{\neg\neg A : A \in \Sigma_1^0\}.$$

Corollary 31.

1. (a) *The following are equivalent:*
LPO; $P(\Sigma_1^0, \Phi)$; $P(\Phi, \Sigma_1^0)$; $P(\Sigma_1^0, \Sigma_1^0)$.
(b) *The following are equivalent:*
WLPO; $P(\Pi_1^0, \Phi)$; $P(\Phi, \Pi_1^0)$; $P(\Pi_1^0, \Pi_1^0)$.
2. (a) LLPO is equivalent to $P(\Sigma_1^0, \Pi_1^0)$.
(b) WLLPO is equivalent to $P(\Pi_1^0, \Xi_1^0)$.

Now let Ψ stand for the class of all negated assertions.

Proposition 32. WLPO, $P(\Sigma_1^0, \Psi)$, and $P(\Sigma_1^0, \Xi_1^0)$ are equivalent.

Proof. In view of $\Xi_1^0 \subseteq \Psi$ we only have to check that

(i) WLPO implies $P(\Sigma_1^0, \Psi)$ and (ii) WLPO follows from $P(\Sigma_1^0, \Xi_1^0)$.
As for (i), let $A \in \Sigma_1^0$ and $F \in \Phi$. Assume that $\neg A \vee \neg\neg A$. Hence if $A \Rightarrow \neg F$ and thus $\neg\neg A \Rightarrow \neg F$, then $\neg A \vee \neg F$. To prove (ii) use $A \Rightarrow \neg\neg A$ for any $A \in \Sigma_1^0$. \square

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References

- [1] Y. Akama, S. Berardi, S. Hayashi, and U. Kohlenbach, An arithmetical hierarchy of the law of excluded middle and related principles. 19th IEEE Symposium on Logic in Computer Science (LICS 2004). Proceedings. IEEE Computer Society, 2004, pp. 192–201.
- [2] E. Bishop, *Foundations of Constructive Analysis*. McGraw–Hill, New York, 1967.
- [3] E. Bishop and D. Bridges, *Constructive Analysis*. Springer, Berlin etc., 1985.
- [4] S. Berardi, *Some intuitionistic equivalents of classical principles for degree 2 formulas*, *Ann. Pure Appl. Logic* **139** (2006), pp. 185–200.
- [5] J. Berger, D. Bridges, and P. Schuster, *The fan theorem and unique existence of maxima*, *J. Symbolic Logic* **71** (2006), pp. 713–720.
- [6] J. Berger and H. Ishihara, *Brouwer’s fan theorem and unique existence in constructive analysis*, *Math. Log. Quart.* **51** (2005), pp. 369–373.
- [7] J. Berger and P. Schuster, *Classifying Dini’s theorem*, *Notre Dame J. Formal Logic* **47** (2006), pp. 253–262.
- [8] D. Bridges and F. Richman, *Varieties of Constructive Mathematics*. Cambridge University Press, 1987.
- [9] D. Bridges and L. Viřã, *Techniques of Constructive Analysis*. Springer, New York, 2006.
- [10] D. van Dalen, *Logic and structure*. 4th ed., Springer, Berlin, 2004.
- [11] R. David, K. Nour, and C. Raffalli, *Introduction à la logique. Théorie de la démonstration*. 2nd ed., Dunod, Paris, 2001.
- [12] N. Gambino and P. Schuster, *Spatiality for formal topologies*, *Math. Structures Comput. Sci.* **17** (2007), pp. 65–80.
- [13] H. Ishihara, *An omniscience principle, the König lemma and the Hahn–Banach theorem*, *Z. Math. Logik Grundlag. Math.* **36** (1990), pp. 237–240.
- [14] H. Ishihara, *Markov’s principle, Church’s thesis and Lindelöf’s theorem*, *Indag. Math. (N.S.)* **4** (1993), pp. 321–325.
- [15] H. Ishihara, *Constructive reverse mathematics: compactness properties*. In: L. Crosilla, P. Schuster, eds., *From Sets and Types to Topology and Analysis*. Oxford Logic Guides 48. Oxford University Press, 2005, pp. 245–267.
- [16] H. Ishihara, *Weak König’s lemma implies Brouwer’s fan theorem: a direct proof*, *Notre Dame J. Formal Logic* **47** (2006), pp. 249–252.
- [17] Kleene, S.C., *Recursive functions and intuitionistic mathematics*. In: L.M. Graves et al., eds., *Proceedings of the International Congress of Mathematicians 1950*. Amer. Math. Soc., Providence, R.I., 1952, pp. 679–685.
- [18] I. Loeb, *Equivalents of the (weak) fan theorem*, *Ann. Pure Appl. Logic* **132** (2005), pp. 51–66.

- [19] I. Loeb, *Indecomposability of \mathbb{R} and $\mathbb{R} \setminus \{0\}$ in constructive reverse mathematics*, Logic J. IGPL **16** (2008), pp. 269–273.
- [20] M. Mandelkern, *Constructively complete finite sets*, Z. Math. Logik Grundlag. Math. **34** (1988), pp. 97–103.
- [21] R. Mines, W. Ruitenburg, and F. Richman, *A Course in Constructive Algebra*. Springer, New York, 1987.
- [22] T. Nemoto, *Determinacy of Wadge classes and subsystems of second order arithmetic*, MLQ Math. Log. Q. **55** (2009), pp. 154–176.
- [23] T. Nemoto, *Complete determinacy and subsystems of second order arithmetic*. In: Logic and theory of algorithms, Lecture Notes in Comput. Sci. 5028, Springer, Berlin, 2008, pp. 457–466.
- [24] T. Nemoto, M. Ould MedSalem, and K. Tanaka, *Infinite games in the Cantor space and subsystems of second order arithmetic*, MLQ Math. Log. Q. **53** (2007), pp. 226–236.
- [25] F. Richman, *The fundamental theorem of algebra: a constructive development without choice*, Pacific J. Math. **196** (2000), pp. 213–230.
- [26] F. Richman, *Constructive mathematics without choice*. In: P. Schuster et al., eds., Reuniting the Antipodes. Constructive and Nonstandard Views of the Continuum. Kluwer, Dordrecht, 2001, pp. 199–205.
- [27] F. Richman, *Spreads and choice in constructive Mathematics*, Indag. Math. (N.S.) **13** (2002), pp. 259–267.
- [28] P. Schuster, *Unique solutions*, Math. Log. Quart. **52** (2006), pp. 534–539. Corrigendum: Math. Log. Quart. **53** (2007), p. 214.
- [29] H. Schwichtenberg, *A direct proof of the equivalence between Brouwer’s fan theorem and König’s lemma with a uniqueness hypothesis*, J. UCS **11** (2005), pp. 2086–2095.
- [30] H. Schwichtenberg and S.S. Wainer, *Proofs and Computations*. Association for Symbolic Logic and Cambridge University Press, 2012.
- [31] S.G. Simpson, *Subsystems of Second Order Arithmetic*, Springer, Berlin etc., 1999.
- [32] M. Toftdal, *A calibration of ineffective theorems of analysis in a hierarchy of semi-classical logical principles*. In: 31st International Colloquium on Automata, Languages and Programming (ICALP 2004). Proceedings. Lecture Notes in Comput. Sci. 3142, Springer, 2004, pp. 1188–1200.
- [33] A.S. Troelstra and D. van Dalen, *Constructivism in Mathematics*. Two volumes. North-Holland, Amsterdam, 1988.
- [34] A. S. Troelstra and H. Schwichtenberg, *Basic Proof Theory*, 2nd edition, Cambridge University Press, 2000.
- [35] W. Veldman, *Brouwer’s fan theorem as an axiom and as a contrast to Kleene’s alternative*. Preprint, Radboud University, Nijmegen, 2005.

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