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THE WEAK KŐNIG LEMMA, BROUWER'S FAN THEOREM, DE MORGAN'S LAW, AND DEPENDENT CHOICE

A b s t r a c t. The standard omniscience principles are interpreted in a systematic way within the context of binary trees. With this dictionary at hand we revisit the weak Kőnig lemma (WKL) and Brouwer's fan theorem (FAN). We first study how one can arrive from FAN at WKL, and then give a direct decomposition, without coding, of WKL into the lesser limited principle of omniscience and an instance of the principle of dependent choices. As a complement we provide, among other equivalents of the standard omniscience principles, a uniform method to formulate most of them.

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1. Introduction

This paper begins with a systematic interpretation in the context of binary trees of the following fragments of the law of excluded middle: the limited, the weak limited, the lesser limited, and the weak lesser limited principle of omniscience; and Markov's principle.

Using this interpretation, we study the interrelations of the following properties, including their negations and double negations, of a binary tree T: T is finite; T is well-founded; T has an infinite path. This enables us to decompose the weak Kőnig lemma into the fan theorem and a higher-type instance of the law of excluded middle.

We further have a closer look at the two fragments of De Morgan's law that occur in this context: the lesser limited and the weak lesser limited principle of omniscience. Among other things, the former principle proves tantamount to a variant of the weak Kőnig lemma.

Next, we consider some forms of countable and dependent choice for disjunctions, and show how the weak Kőnig lemma can be decomposed into such a form of dependent choice plus the lesser limited principle of omniscience. Finally, we provide a uniform method to formulate all but one of the aforementioned fragments of the law of excluded middle: as the implication from $A \Rightarrow B$ to $\neg A \vee B$ restricted to certain classes of assertions A, B.

This paper is a contribution to the constructive variant, put forward as a programme in [15, 16], of "reverse mathematics" [31].[1] An intuitionistic counterpart was started in parallel [18, 35], and there also is a "refined intuitionistic reverse mathematics" [1, 4, 32].

We proceed in Bishop's constructive mathematics [2, 3, 8, 9], the principal characteristic of which is the exclusive use of intuitionistic logic.[2] Since we are only concerned with basic operations on natural numbers and binary strings, our results can be carried over to a formal system based on intuitionistic analysis **EL** [33, 3.6] or on intuitionistic finite-type arithmetic $\mathbf{HA}^\omega$ [33, 9.1] enriched with quantifier-free number-number choice. As in those two systems, we assume that our universe of functions (i.e., sequences) of natural numbers contains all the primitive recursive (or elementary) functions, and is closed under composition and primitive re-

[1] Where also infinite games and related issues have been studies, see e.g. [22, 23, 24].

[2] For introductions to intuitionistic logic see, for instance, [10, 11, 34, 33, 30].

cursion; in particular, our universe of functions is closed under bounded minimisation. Following [25, 26, 27] we work in Bishop-style constructive mathematics without any form of countable choice that is not derivable from those assumptions.

2. Preliminaries

By $m, n, k, \ldots$ we always mean elements of the set $\mathbb{N} = \{0, 1, 2, \ldots\}$. It is sometimes useful to follow a set-theoretic tradition and identify each $n \in \mathbb{N}$ with the set $\{0, \ldots, n-1\}$ of its predecessors.

A (finite or infinite) *binary sequence* is a (finite or infinite) sequence of elements of $\{0, 1\}$. We use $\alpha, \beta, \gamma, \ldots$ and $u, v, w, \ldots$ as variables for the infinite and finite binary sequences, respectively: that is, for the elements of the sets $\{0, 1\}^{\mathbb{N}}$ and

$$\{0,1\}^{*} = \bigcup \{\{0,1\}^{n} : n \geqslant 0\} \ .$$

We further denote the *length* of u by $|u|$; in other words,

$$|u| = n \Leftrightarrow u \in \{0,1\}^{n} \ .$$

The *n-th finite initial segment*

$$\overline{\alpha}n = (\alpha(0), \ldots, \alpha(n-1))$$

of α has length n. In particular, $\overline{\alpha}0$ is the *empty sequence* $()$, the only one of length 0. Likewise,

$$\overline{u}k = (u(0), \ldots, u(k-1))$$

is defined for every $k \leqslant |u|$. Note that $\overline{u}|u| = u$, and that $\overline{\overline{\alpha}n}k = \overline{\alpha}k$ whenever $k \leqslant n$.

We identify each binary sequence (a) of length 1 with its only element a, and denote the *concatenation* of u and v by their juxtaposition uv. If $ui = w$ for some $i \in \{0, 1\}$, then w is an *immediate successor* of u. We write $u \leqslant w$ if u is a *restriction* of w: that is,

$$u \leqslant w \Leftrightarrow \exists v \ (uv = w) \ .$$

An assertion $D(x)$ about the elements x of a set X is *decidable* if there is a function $f : X \to \mathbb{N}$ such that $D(x)$ is equivalent to $f(x) = 0$ for each $x \in X$. A subset S of a set X is *detachable* if $x \in S$ is a decidable assertion about the elements x of X. In other words, a detachable subset can be identified with its characteristic function.

An assertion $A(x)$ about the elements x of a set X is

— *simply existential* if it has the form $\exists n\, D(x, n)$ with $D(x, n)$ a decidable assertion;

— *simply universal* if it has the form $\forall n\, D(x, n)$ with $D(x, n)$ a decidable assertion.

(We have seen the notion of a simply existential assertion first in [8], and both are used in [1].) We sometimes write Δ_0, Σ^0_1, and Π^0_1 for the classes of the decidable, the simply existential, and the simply universal assertions, respectively. Note that finite disjunctions (respectively, finite conjunctions) of simply existential (respectively, simply universal) assertions are simply existential (respectively, simply universal).

We next study the versions of *Countable Choice* ($\mathrm{CC}^\vee$) and of *Dependent Choice* ($\mathrm{DC}^\vee$) for disjunctions [15, 16.5]:

$\mathbf{CC}^\vee$ $\forall n\, (A_0(n) \vee A_1(n)) \Rightarrow \exists \alpha\, \forall n\, A_{\alpha(n)}(n)\,;$

$\mathbf{DC}^\vee$ $\forall u\, (A_0(u) \vee A_1(u)) \Rightarrow \exists \alpha\, \forall n\, A_{\alpha(n)}(\overline{\alpha}n)\,.$

Proposition 1. $\mathrm{DC}^\vee$ *implies* $\mathrm{CC}^\vee$.

Proof. If $\forall n\, (A_0(n) \vee A_1(n))$, then $\forall u\, (A_0(|u|) \vee A_1(|u|))$; whence, by $\mathrm{DC}^\vee$, there is α with $\forall n\, A_{\alpha(n)}(|\overline{\alpha}n|)$ or, equivalently, $\exists \alpha\, \forall n\, A_{\alpha(n)}(n)$. □

By Γ-$\mathrm{CC}^\vee$ (respectively, by Γ-$\mathrm{DC}^\vee$) we denote $\mathrm{CC}^\vee$ (respectively, $\mathrm{DC}^\vee$) as restricted to an assertion class Γ:

Γ-CC$^\vee$ *is* $\mathrm{CC}^\vee$ *for all assertions* $A_0(n)$ *and* $A_1(n)$ *which belong to* Γ *;*

Γ-DC$^\vee$ *is* $\mathrm{DC}^\vee$ *for all assertions* $A_0(u)$ *and* $A_1(u)$ *which belong to* Γ *.*

By inspection of the proof of Proposition 1 one can prove the following:

Corollary 2. Γ-$\mathrm{DC}^\vee$ *implies* Γ-$\mathrm{CC}^\vee$ *whenever* Γ *is* Δ_0*,* Σ^0_1*, or* Π^0_1*.*

Proposition 3. Δ_0-DC$^\vee$ *and* Δ_0-CC$^\vee$ *are provable.*

Proof. In view of Corollary 2 it suffices to prove Δ_0-DC$^\vee$.

Let $\forall u\ (A_0(u) \vee A_1(u))$. By primitive recursion, one can construct the leftmost possible path α with $\forall n\, A_{\alpha(n)}(\overline{\alpha}n)$: simply set

$$\alpha\,(n) = \min\left\{i \in \{0,1\} : A_i\,(\overline{\alpha}n)\right\}$$

for every n. □

To prove the converse of Proposition 1 we need—in addition to Δ_0-DC$^\vee$—the following *coding*, which holds for the supposed presence of all primitive recursive functions:

CODE *There are functions* $f : \mathbb{N} \to \{0,1\}^*$ *and* $g : \{0,1\}^* \to \mathbb{N}$ *with* $f \circ g = \mathrm{id}_{\{0,1\}^*}$.

Proposition 4. CC$^\vee$ *implies* DC$^\vee$.

Proof. Assume that f and g are as in CODE. If $\forall u\ (A_0(u) \vee A_1(u))$, then, in particular, $\forall n\ (A_0\,(f(n)) \vee A_1(f(n)))$; whence by CC$^\vee$ there is β with $\forall n\, A_{\beta(n)}(f(n))$. Since $\forall u\ (\beta\,(g(u)) = 0 \vee \beta\,(g(u)) = 1)$, by Δ_0-DC$^\vee$ there exists α such that $\forall n\ (\beta\,(g(\overline{\alpha}n)) = \alpha(n))$. For this α we have $\forall n\, A_{\beta(g(\overline{\alpha}n))}(f(g(\overline{\alpha}n)))$ or, equivalently, $\forall n\, A_{\alpha(n)}(\overline{\alpha}n)$. □

As for Proposition 1 and Corollary 2, a close look at the proof of Proposition 4 shows that this implication, too, can be relativised (recall that a decidable assertion is identified with its characteristic function):

Corollary 5. Γ-CC$^\vee$ *implies* Γ-DC$^\vee$ *whenever* Γ *is* Δ_0, Σ^0_1, *or* Π^0_1.

3. Binary trees and omniscience principles

The *Limited Principle of Omniscience* (LPO), the *Weak Limited Principle of Omniscience* (WLPO), and *Markov's Principle* (MP) read as follows [8]:

LPO $A \vee \neg A$ *for all simply existential assertions* A;

WLPO $\neg A \vee \neg\neg A$ *for all simply existential assertions* A;

MP $\neg\neg A \Rightarrow A$ *for all simply existential assertions* A.

It is plain that LPO is equivalent to the conjunction of WLPO and MP. The negations of the simply existential assertions are precisely the simply universal assertions. In particular, WLPO is equivalent to

$$B \vee \neg B \text{ for all simply universal assertions } B\,,$$

the counterpart of LPO for simply universal assertions rather than simply existential ones. (The corresponding counterpart of MP is of no interest for it is generally valid: if A is simply universal, then A is of the form $\neg B$ with B simply existential, and $\neg\neg A \Rightarrow A$ is nothing but $\neg\neg\neg B \Rightarrow \neg B$, which holds for every B.)

A subset T of $\{0,1\}^*$ is *closed under restrictions* if

$$u \leqslant w \wedge w \in T \Rightarrow u \in T\ ;$$

any such T is inhabited precisely when $() \in T$. A *tree* is an inhabited, detachable subset T of $\{0,1\}^*$ that is closed under restrictions.

A tree T is

— *finite* if $\exists n\, \forall u\ (|u| = n \Rightarrow u \notin T)$,

— *infinite* if $\forall n\, \exists u\ (|u| = n \wedge u \in T)$.

A tree T is finite if and only if it is a finite set in the usual sense that there is a uniquely determined $N \in \mathbb{N}$, the cardinality of T, for which there is a bijection between $\{1, \ldots, N\}$ and T.[3] (In fact $N > 0$ for T is inhabited.) Moreover, a tree is infinite if and only if it is an infinite set according to any of the following two equivalent definitions:

(a) for every finite subset F of T there is $u \in T$ with $u \notin F$;

(b) for every $N \in \mathbb{N}$ there is a subset of T that has cardinality N.

[3]Since $\{0,1\}^*$ is a discrete set (that is, $u = v \vee u \neq v$ for all u, v), for subsets T of $\{0,1\}^*$ there is no need to distinguish "finite" from "finitely enumerable" [8, 21] (or "subfinite" [2, 3]), which means the existence of a—not necessarily uniquely determined—$N \in \mathbb{N}$ together with a mapping from $\{1, \ldots, N\}$ onto T. In fact, if a discrete set is finitely enumerable, then it is finite.

(To arrive from (a) at (b) requires induction.) Clearly, (b) follows from

(c) there is an infinite sequence of mutually distinct elements of T,

whereas by quantifier-free number-number choice together with CODE and primitive recursion one can show that (a) implies (c).

Throughout this paper, the first and easy lemma will be used without mention.

Lemma 6. *If T is a tree, then T is infinite if and only if it is not finite, and the assertions "T finite" and "T infinite" are simply existential and simply universal, respectively.*

Proof. Let T be a tree. It suffices to observe that

$$\forall u\ (|u| = n \Rightarrow u \notin T) \quad \text{and} \quad \exists u\ (|u| = n \wedge u \in T)$$

are decidable assertions about n each of which is the negation of the other. $\square$

If the simply existential assertion A is of the form $\exists n\, D(n)$ with $D(n)$ a decidable assertion, then

$$T[A] = \{()\} \cup \bigcup \left\{ \{0,1\}^{n+1} : n \geqslant 0 \wedge \neg D(0) \wedge \ldots \wedge \neg D(n) \right\}$$

is a tree with

$$A \Leftrightarrow \exists n \left(T[A] = \bigcup \left\{ \{0,1\}^k : 0 \leqslant k \leqslant n \right\} \right) \quad \text{and} \quad \neg A \Leftrightarrow T[A] = \{0,1\}^* .$$

Lemma 7. *For every simply existential assertion A the tree $T[A]$ is such that A (respectively, $\neg A$) holds if and only if $T[A]$ is finite (respectively, $T[A]$ is infinite).*

In all, we have the following proposition.

Proposition 8.

1. LPO *is equivalent to "every tree is either finite or infinite".*
2. WLPO *is equivalent to "every tree is either infinite or not infinite".*
3. MP *is equivalent to "if a tree is not infinite, then it is finite".*

A statement similar to the last item of the preceding proposition can be found in [4, Lemma 3.2]: the counterpart of MP on type level 2, i.e. the stability of all Σ_2^0-formulas, is equivalent to "if a recursive set of integers is not infinite, then it is finite".

A tree T

— is *well-founded* [33, 4.8.1] if $\forall \alpha \, \exists n \, (\overline{\alpha} n \notin T)$;

— has an *infinite path* if $\exists \alpha \, \forall n \, (\overline{\alpha} n \in T)$.

Each of the properties "T is well-founded" and "T has an infinite path" clearly entails the negation of the other. The reverse implications will occur in the sequel.

Lemma 9. *For every simply existential assertion A the tree $T[A]$ is such that A (respectively, $\neg A$) holds if and only if $T[A]$ is well-founded (respectively, $T[A]$ has an infinite path).*

It is in order to indicate that to prove the implication "if $T[A]$ is well-founded, then A holds" (in fact, to prove "if $T[A]$ is well-founded, then $T[A]$ is finite", see Lemma 7) requires to have at least one infinite path at one's disposal. The assumptions we have made about our universe of functions ensure, for instance, the presence of the path which is constantly zero.

Proposition 10. MP *is equivalent to the following statement:*

($*$) *If a tree has no infinite path, then this tree is well-founded.*

Proof. Assume first MP, and let T be a tree. By MP we have

$$\forall \alpha \, (\neg \forall n \, (\overline{\alpha} n \in T) \Rightarrow \exists n \, (\overline{\alpha} n \notin T)) \ ,$$

from which

$$\forall \alpha \, \neg \forall n \, (\overline{\alpha} n \in T) \Rightarrow \forall \alpha \, \exists n \, (\overline{\alpha} n \notin T)$$

follows—or, equivalently,

$$\neg\exists\alpha\,\forall n\ (\overline{\alpha}n \in T) \Rightarrow \forall\alpha\,\exists n\ (\overline{\alpha}n \notin T)\ ,$$

which is nothing but $(*)$ for the tree T under consideration.

To deduce MP from $(*)$, let A be a simply existential assertion, and set $T = T[A]$. By Lemma 9, $\neg\neg A$ amounts to "T has no infinite path". By $(*)$, this implies that T is well-founded, which is equivalent to A again by Lemma 9. □

4. The weak Kőnig lemma and Brouwer's fan theorem

If a tree has an infinite path, then this tree is infinite; if a tree is finite, then it is well-founded. The reverse implications are known as the *Weak Kőnig Lemma* (WKL) and as (an equivalent [7, 16] of) *Brouwer's Fan Theorem* (FAN)*:*

WKL *Every infinite tree has an infinite path;*

FAN *Every well-founded tree is finite.*

The contrapositives of WKL and FAN read as follows:

WKL′ *If a tree has no infinite path, then this tree is not infinite;*

FAN′ *If a tree is infinite, then this tree is not well-founded.*

Note in this context that every implication $A \Rightarrow B$ entails its contrapositive $\neg B \Rightarrow \neg A$, which is equivalent to all the iterated contrapositives. In fact, $\neg\neg A \Rightarrow \neg\neg B$ implies $\neg\neg\neg B \Rightarrow \neg\neg\neg A$, which is an equivalent of $\neg B \Rightarrow \neg A$.

Some of the next results have counterparts in [12, Proposition 3.3].

Lemma 11.

1. WKL′ *is equivalent to the following statement:*

   If a tree is infinite, then it cannot fail to have an infinite path.

2. FAN′ *is equivalent to the following statement:*

If a tree is well-founded, then it is not infinite.

Proof. The following are equivalent for arbitrary assertions A and B: $A \Rightarrow \neg B$; $\neg(A \wedge B)$; $B \Rightarrow \neg A$. □

It is well known (see, e.g., [16]) that WKL implies FAN in the presence of MP. This can be sharpened:

Corollary 12. *The following items are equivalent:*

1. $\mathrm{WKL}' + \mathrm{MP}$;
2. *If a tree has no infinite path, then it is finite;*
3. $\mathrm{FAN} + \mathrm{MP}$.
4. $\mathrm{FAN}' + \mathrm{MP}$.

Proof. The equivalence of items 1 and 2 and of items 2 and 3 is readily seen if one understands MP as characterised in Proposition 8 and in Proposition 10, respectively. Using the former characterisation of MP and the equivalent of FAN′ given in Lemma 11 it is plain that items 3 and 4 are equivalent. □

It can be proved without using MP that WKL implies FAN [13, 16]. In fact, FAN is equivalent to WKL as restricted to trees with, in an appropriate sense, at most one infinite path [6, 29] (see [5, 28] for related results). Since WKL′ implies FAN′, we thus have the following situation:

$$\begin{array}{ccc} \mathrm{WKL} & \Rightarrow & \mathrm{WKL}' \\ \Downarrow & & \Downarrow \\ \mathrm{FAN} & \Rightarrow & \mathrm{FAN}' \end{array}$$

In the sequel we will discuss the following statements: first,

($\flat$) *If a tree is not well-founded, then it cannot fail to have an infinite path,*

which is the contrapositive of ($*$) and equivalent to

If a tree has no infinite path, then it cannot fail to be well-founded;

next, the counterpart of $(*)$ with "has an infinite path" and "is well-founded" interchanged:

(†) *If a tree is not well-founded, then this tree has an infinite path;*

and, finally, the stability of "has an infinite path" and "is well-founded":

(§) *If a tree cannot fail to have an infinite path, then it has an infinite path;*

(♮) *If a tree cannot fail to be well-founded, then it is well-founded.*

Lemma 13.

1. *The conjunction of* $(\flat)$ *and* (§) *is equivalent to* (†).
2. *The conjunction of* $(\flat)$ *and* $(\natural)$ *is equivalent to* $(*)$.

Proof. If $A \Rightarrow \neg B$, then $\neg\neg B \Rightarrow \neg A$; whence the following are equivalent:

$$\neg A \Rightarrow B;\ \ (\neg B \Rightarrow \neg\neg A) \wedge (\neg\neg B \Rightarrow B);\quad (\neg A \Rightarrow \neg\neg B) \wedge (\neg\neg B \Rightarrow B)\,.$$

□

Proposition 14.

1. $\mathrm{WKL} \Leftrightarrow \mathrm{WKL}' + (\S)\,;$
2. $\mathrm{WKL}' \Leftrightarrow \mathrm{FAN}' + (\flat)\ ;$
3. $\mathrm{WKL} \Leftrightarrow \mathrm{FAN}' + (\dagger)\,.$

Proof. In view of Lemma 13, part 3 follows from parts 1 and 2, which are readily shown with Lemma 11 at hand. □

Corollary 15.

1. $\mathrm{WKL} \Leftrightarrow \mathrm{FAN} + (\dagger)\,.$
2. *Under the assumption of* (†) *the following are equivalent:* WKL*;* WKL′*;* FAN′*;* FAN.

5. Fragments of De Morgan's law

We first recall the *Lesser Limited Principle of Omniscience* (LLPO):

LLPO $\neg(A_0 \wedge A_1) \Rightarrow \neg A_0 \vee \neg A_1$ *for all simply existential assertions* A_0, A_1.

Lemma 16. LLPO *is equivalent to the statement that*

$$\forall u \neg (A_0(u) \wedge A_1(u)) \Rightarrow \forall u\ (\neg A_0(u) \vee \neg A_1(u)) \tag{1}$$

holds for all assertions $A_0(u)$ *and* $A_1(u)$ *which are simply existential for every* u.

Proof. It is clear that LLPO implies

$$\forall u\ (\neg (A_0(u) \wedge A_1(u)) \Rightarrow \neg A_0(u) \vee \neg A_1(u)),$$

from which (1) follows. As for the converse, apply (1) to $A_i(u) \equiv A_i$. □

The following is [8, Chapter 1, Problem 1]; we give a proof for completeness's sake, and refer to [1, Theorem 3.14] for the underivability of WLPO from LLPO in HA.

Lemma 17. WLPO *implies* LLPO.

Proof. Let A_0 and A_1 be simply existential assertions, and assume $\neg(A_0 \wedge A_1)$. By WLPO both $\neg A_0 \vee \neg\neg A_0$ and $\neg A_1 \vee \neg\neg A_1$: that is, either $\neg A_0 \vee \neg A_1$ or else $\neg\neg A_0 \wedge \neg\neg A_1$. The latter alternative is equivalent to $\neg\neg(A_0 \wedge A_1)$, which contradicts $\neg(A_0 \wedge A_1)$. □

If T is a detachable subset of $\{0,1\}^*$, then so are

$$uT = \{uv : v \in T\} \quad \text{and} \quad T_u = \{v : uv \in T\}$$

for every u. Note that $(uT)_u = T$ and $u(T_u) \subseteq T$, and that $()\,T = T$ and $T_{()} = T$. If T is closed under restrictions, then so is T_u; moreover, $u \in T$ precisely when $() \in T_u$. In all, if T is a tree, then T_u is a tree—the "subtree of T with root u"—precisely when $u \in T$.

Lemma 18. *Any* α *is an infinite path in a tree* T *precisely when* $T_{\overline{\alpha}n}$ *is infinite for all* n.

Proof. If $\overline{\alpha}k \in T$ for all $k \geqslant n$, then $T_{\overline{\alpha}n}$ is infinite. Conversely, if $T_{\overline{\alpha}n}$ is infinite, then $() \in T_{\overline{\alpha}n}$ or, equivalently, $\overline{\alpha}n \in T$. □

Let T be a tree. Since $T_u \setminus \{()\} = 0T_{u0} \cup 1T_{u1}$, we have

$$\begin{aligned} T_u \text{ finite } &\Leftrightarrow\ T_{u0} \text{ finite } \wedge\ T_{u1} \text{ finite}\,, \\ T_u \text{ infinite } &\Leftrightarrow\ \neg\,(T_{u0} \text{ finite } \wedge\ T_{u1} \text{ finite})\ . \end{aligned} \tag{2}$$

Proposition 19. LLPO *is equivalent to the statement that, for every tree T,*

$$T \textit{ infinite } \Rightarrow\ T_0 \textit{ infinite } \vee\ T_1 \textit{ infinite}. \tag{3}$$

Proof. By (2), LLPO implies (3) for every tree T. As for the converse, let A_0 and A_1 be simply existential assertions. Consider the tree

$$T = \{()\} \cup 0T[A_0] \cup 1T[A_1]\ .$$

For every $i \in \{0, 1\}$ we have $T_i = T[A_i]$ and thus, in view of Lemma 7,

$$T_i \text{ finite} \Leftrightarrow A_i \quad \text{and} \quad T_i \text{ infinite} \Leftrightarrow \neg A\,.$$

By (2), the required instance of LLPO follows from (3). □

Corollary 20. WKL *implies* LLPO.

Proof. Assume WKL. To show LLPO as characterised by Proposition 19, let T be an infinite tree. By WKL, T has an infinite path α. Hence $T_{\alpha(0)} = T_{\overline{\alpha}1}$ is infinite (Lemma 18). □

The following principle similar to (3) has occurred with parameters in [4, p. 189].

1-Kőnig For all decidable assertions $D_i(n_i)$ with $i \in \{0, 1\}$,

$$\forall n_0 \forall n_1\, (D_0(n_0) \vee D_1(n_1)) \Rightarrow \forall n_0\, D_0(n_0) \vee \forall n_1\, D_1(n_1)\,. \tag{4}$$

The implication from LLPO to 1-Kőnig was indicated in the proofs of [1, Theorems 2.7, 3.1]. Now if $D_i(n_i)$ is a decidable assertion for each $i \in \{0, 1\}$, and $A_i \equiv \exists n_i\, \neg D_i(n_i)$, then

$$\begin{aligned} \neg\,(A_0 \wedge A_1)\ &\Longleftrightarrow\ \forall n_0 \forall n_1\, (D_0(n_0) \vee D_1(n_1))\ , \\ \neg A_0 \vee \neg A_1\ &\Longleftrightarrow\ \forall n_0 D_0(n_0)\ \vee\ \forall n_1 D_1(n_1)\,. \end{aligned}$$

Hence we have the following:

Proposition 21. LLPO *and* 1-Kőnig *are equivalent.*

We next formulate the *Weak Lesser Limited Principle of Omniscience* (WLLPO):

WLLPO $\neg(\neg A_0 \wedge \neg A_1) \Rightarrow \neg\neg A_0 \vee \neg\neg A_1$ *for all simply existential assertions* A_0, A_1.

For proofs that LLPO implies WLLPO using real numbers and binary sequences see [20, Theorem 4.1, Theorem 4.2] and [14, Proposition 1.2], respectively. While WLLPO occurs as LLPE in the former reference, in the latter LLPO and WLLPO are called SEP and $\mathrm{MP}^{\vee}$, respectively. Our choice of the name for WLLPO is motivated by the fact that WLLPO is to LLPO just as WLPO is to LPO. More precisely, WLLPO is tantamount to

$$\neg(B_0 \wedge B_1) \Rightarrow \neg B_0 \vee \neg B_1 \text{ for all simply universal assertions } B_0, B_1,$$

the counterpart of LLPO for simply universal assertions rather than simply existential ones. Note also that WLLPO is equivalent to the statement that, for every tree T,

$$\neg(T_0 \text{ infinite } \wedge\ T_1 \text{ infinite}) \ \Rightarrow\ T_0 \text{ not infinite } \vee\ T_1 \text{ not infinite},$$

just as LLPO was characterised in Proposition 19.

Furthermore, WLLPO follows from MP, which is called LPE in [20]. More precisely, MP is equivalent to WLLPO in conjunction with the weak Markov principle WMP [14, Proposition 1.1]—or, in other terms, LPE is equivalent to LLPE plus WLPE [20, Section 1]. Also, WMP is a consequence of a form of Church's thesis for disjunctions, under which it thus is equivalent to MP [14, Proposition 2, Theorem 1].

Here is another approach to the implications from LLPO and from MP to WLLPO. We say that a decidable assertion $D(n)$ *holds for at most one* n if $m = n$ whenever $D(m)$ and $D(n)$. It is well-known that LLPO and MP can equivalently put as follows:

LLPO$_0$ *If a decidable assertion* $D(n)$ *holds for at most one* n*, then there is* $i \in \{0,1\}$ *with* $\forall k\, \neg D(2k+i)$.

MP$_0$ *For every decidable assertion $D(n)$ which holds for at most one n, if $\neg\forall n\,\neg D(n)$, then there is $i \in \{0,1\}$ with $\exists k\, D(2k+i)$.*

A similar characterisation of WLLPO was inspired by the proof of [14, Proposition 1] and by a remark in [19, Section 3]:[4]

WLLPO$_0$ *For every decidable assertion $D(n)$ which holds for at most one n, if $\neg\forall n\,\neg D(n)$, then there is $i \in \{0,1\}$ with $\forall k\,\neg D(2k+i)$.*

In MP$_0$ and WLLPO$_0$ the index $i \in \{0,1\}$ is uniquely determined by the required property.

Lemma 22. WLLPO *and* WLLPO$_0$ *are equivalent.*

Proof. For each $i \in \{0,1\}$ set $\bar{\imath} = 1 - i$. Assume first WLLPO, and let $D(n)$ be a decidable assertion. For each $i \in \{0,1\}$ set

$$A_i \equiv \exists\, k\, D(2k+i)\,,$$

for which $\neg A_0 \wedge \neg A_1$ is equivalent to $\forall n\,\neg D(n)$. If $\neg\forall n\,\neg D(n)$, which is to say that $\neg(\neg A_0 \wedge \neg A_1)$, then $\neg\neg A_0 \vee \neg\neg A_1$ by WLLPO. If $D(n)$ holds for at most one n, then $\neg A_{\bar{\imath}}$ follows from A_i, and thus already from $\neg\neg A_i$.

Assume next WLLPO$_0$, and let A_i be a simply existential assertion of the form

$$A_i \equiv \exists k\, D_i(k)$$

where $D_i(k)$ is a decidable assertion for each $i \in \{0,1\}$. Define decidable assertions $E(n)$ and $F(n)$ by

$$E(2k+i) \equiv D_i(k)$$

for every k and each $i \in \{0,1\}$, and by

$$F(n) \equiv E(n) \wedge \forall m < n\, \neg E(m)$$

[4] This remark says that the following statements are equivalent:

$$\forall x \in \mathbb{R}\; (\neg(x \leqslant 0 \wedge x \geqslant 0) \Rightarrow \neg(x \leqslant 0) \vee \neg(x \geqslant 0))\;;$$
$$\forall x \in \mathbb{R}\; (\neg(x \leqslant 0 \wedge x \geqslant 0) \Rightarrow x \leqslant 0 \vee x \geqslant 0)\;.$$

Note in this context that $x \leqslant y$ is a simply universal statement for every pair $x, y \in \mathbb{R}$, for it is the negation of the simply existential statement $x > y$.

for every n. Note that $F(n)$ holds for at most one n, and that A_i is equivalent to $\exists k\, E(2k+i)$. In particular, $\neg A_0 \wedge \neg A_1$ is equivalent to $\forall n\, \neg E(n)$, and $\exists k\, F(2k+i)$ implies A_i. Moreover, $\exists n\, E(n)$ implies $\exists n\, F(n)$; whence $\forall n\, \neg F(n)$ implies $\forall n\, \neg E(n)$. Now if $\neg(\neg A_0 \wedge \neg A_1)$, then $\neg \forall n\, \neg F(n)$; whence if $\forall k\, \neg F(2k+i)$, then $\neg \forall k\, \neg F(2k+\bar{\imath})$, and thus $\neg\neg A_{\bar{\imath}}$ for each $i \in \{0,1\}$. □

With LLPO and MP as LLPO_0 and MP_0, respectively, the following is evident:

Corollary 23. *Each of* LLPO *and* MP *implies* WLLPO.

Proposition 24. *Each of the following statements is equivalent to* WLLPO*:*

1. $\neg(\neg A_0 \wedge \neg A_1) \wedge \neg(A_0 \wedge A_1) \Rightarrow \neg\neg A_0 \vee \neg\neg A_1$ *for all simply existential assertions* A_0, A_1*;*
2. $\neg(\neg A_0 \wedge \neg A_1) \wedge \neg(A_0 \wedge A_1) \Rightarrow (\neg\neg A_0 \vee \neg\neg A_1) \wedge (\neg A_1 \vee \neg A_0)$ *for all simply existential assertions* A_0, A_1*;*
3. $\neg(\neg A_0 \wedge \neg A_1) \wedge \neg(A_0 \wedge A_1) \Rightarrow \neg A_1 \vee \neg A_0$ *for all simply existential assertions* A_0, A_1.

Proof. The following are equivalent: $\neg(B \wedge C)$; $B \Rightarrow \neg C$; $\neg\neg B \Rightarrow \neg C$. In particular, items 1, 2, and 3 are equivalent. It is obvious that WLLPO implies item 1, whereas WLLPO as characterised in Lemma 22 is readily deduced from item 3: apply 3 to $A_i \equiv \exists\, k\, D(2k+i)$ whenever $\mathrm{D}(n)$ is a decidable assertion. □

In view of its equivalent MP_0, it is clear that MP is tantamount to

$$\neg(\neg A_0 \wedge \neg A_1) \Rightarrow A_0 \vee A_1 \text{ for all simply existential assertions } A_0,\, A_1\,.$$

In particular, MP is equivalent to the statement that, for every tree T,

$$\neg(T_0 \text{ infinite } \wedge\ T_1 \text{ infinite}) \;\Rightarrow\; T_0 \text{ finite } \vee\ T_1 \text{ finite}\,.$$

Moreover, one can characterise MP just as WLLPO was treated in Proposition 24:

Proposition 25. *Each of the following statements is equivalent to* MP*:*

1. $\neg(\neg A_0 \wedge \neg A_1) \wedge \neg(A_0 \wedge A_1) \Rightarrow A_0 \vee A_1$ *for all simply existential assertions* A_0, A_1*;*

2. $\neg(\neg A_0 \wedge \neg A_1) \wedge \neg(A_0 \wedge A_1) \Rightarrow (A_0 \vee A_1) \wedge (\neg A_1 \vee \neg A_0)$ *for all simply existential assertions* A_0, A_1*.*

6. The weak Kőnig lemma and dependent choice

In [13], WKL and LLPO were shown to be equivalent. While no other principle was used for deducing LLPO from WKL (see also our Corollary 20), dependent choice was taken for granted for proving the implication from LLPO to WKL. In [1, 2.2] it was pointed out that LLPO is equivalent to WKL with respect to a binary choice for simply universal formulas. Also using dependent choice, a deduction of WKL from 1-Kőnig—which is an equivalent of LLPO (see Proposition 21 above and the discussion preceding it)—was given in [4, Footnote 2].

Following [15, 16.5], we next decompose WKL into LLPO and a weak form of dependent choice. The following equivalence, with LLPO named Σ^0_1-DML, was proved in [15, 16.5]:

$$\mathrm{WKL} \Leftrightarrow \mathrm{LLPO} + \Pi^0_1\text{-}\mathrm{CC}^\vee . \tag{5}$$

The proof of (5) given in [15, 16.5] goes through a third equivalent of a somewhat topological character and requires some coding. This can be avoided, as follows, if one uses Π^0_1-DC$^\vee$ in place of Π^0_1-CC$^\vee$. The proofs of the next two results are unwindings of the proof of [31, Lemma IV.4.4].

Theorem 26. WKL *implies* Π^0_1-DC$^\vee$.

Proof. Assume WKL. To prove Π^0_1-DC$^\vee$ let $A_0(u)$ and $A_1(u)$ be simply universal for every u: that is, A_i is of the form $\forall k\, D_i(u,k)$ where $D_i(u,k)$ is a decidable assertion for each $i \in \{0,1\}$. Set

$$T = \left\{ u : \forall n < |u|\, \forall k < |u|\, D_{u(n)}(\overline{u}n, k) \right\} ,$$

which clearly is a tree. For each α we have

$$\forall m\, \forall n < m\, \forall k < m\, D_{\alpha(n)}(\overline{\alpha}n, k) \Leftrightarrow \forall n \forall k D_{\alpha(n)}(\overline{\alpha}n, k)$$

and thus

$$\forall m\ (\overline{\alpha}m \in T) \Leftrightarrow \forall n\, A_{\alpha(n)}(\overline{\alpha}n)\,.$$

In other words, an infinite path in T is nothing but an infinite sequence α as in the conclusion of $\mathrm{DC}^{\vee}$. By WKL it therefore suffices to show that T is infinite whenever the hypothesis of $\mathrm{DC}^{\vee}$ holds. To this end, consider

$$S = \left\{u : \forall n < |u|\, A_{u(n)}\left(\overline{u}n\right)\right\} = \left\{u : \forall n < |u|\, \forall k\, D_{u(n)}\left(\overline{u}n, k\right)\right\}\ ,$$

and observe that $S \subseteq T$. Now if $A_i(u)$, then $ui \in S$; whence if $\forall u\,(A_0(u) \vee A_1(u))$, then

$$\forall u\ (u \in S \Rightarrow u0 \in S \vee u1 \in S)\ .$$

By induction, for every m there is u with $|u| = m$ such that $u \in S$ and thus $u \in T$. □

Theorem 27. WKL *follows from* LLPO+Π^0_1-$\mathrm{DC}^{\vee}$*.*

Proof. Assume LLPO and Π^0_1-$\mathrm{DC}^{\vee}$. To deduce WKL, let T be a tree and set

$$D\,(m, u) \equiv \exists v\ (|v| = m \wedge v \in T_u)\ .$$

This is a decidable assertion and satisfies

$$\forall m\, D\,(m, u) \Leftrightarrow T_u \text{ infinite}$$

for every u. For each $i \in \{0, 1\}$ we set $\overline{\imath} = 1 - i$ and define

$$A_i\,(u) \equiv \exists m\,(\neg D\,(m, ui) \wedge D\,(m, u\overline{\imath}))\ ,$$

which is simply existential for every u.

We next show that $\forall u\, \neg(A_0(u) \wedge A_1(u))$. Assume that $A_0(u) \wedge A_1(u)$, which is to say that there are m_0 and m_1 with $\neg D\,(m_0, u0)$, $D\,(m_0, u1)$, $\neg D\,(m_1, u1)$, and $D\,(m_1, u0)$. Suppose that $m_0 \geqslant m_1$. By $D\,(m_0, u1)$ there is $v \in T_{u1}$ with $|v| = m_0$, so that for $w = \overline{v}m_1$ we have $w \in T_{u1}$ and $|w| = m_1$ in contradiction to $\neg D\,(m_1, u1)$. The case $m_0 \leqslant m_1$ can be treated in the same way, using first $D\,(m_1, u0)$ and then $\neg D\,(m_0, u0)$.

By LLPO as characterised in Lemma 16 we thus have $\forall u\,(\neg A_0(u) \vee \neg A_1(u))$; whence by Π^0_1-$\mathrm{DC}^{\vee}$ there is α such that $\forall n\, \neg A_{\alpha(n)}(\overline{\alpha}n)$. In view of Lemma 18, it now suffices to show that if T is infinite, then $T_{\overline{\alpha}n}$ is infinite for this α and every n. We proceed by induction on n, using that $T_{\overline{\alpha}n}$ is infinite precisely when $\forall m\, D\,(m, \overline{\alpha}n)$.

The case $n = 0$ amounts to T being infinite.

To deduce $\forall m\, D\,(m, \overline{\alpha}\,(n+1))$ from $\forall m\, D\,(m, \overline{\alpha}n)$, suppose the latter, and fix an arbitrary m. By the very definition of $D\,(m+1, \overline{\alpha}n)$ we have $D\,(m, (\overline{\alpha}n)\,0)$ or $D\,(m, (\overline{\alpha}n)\,1)$. If both alternatives hold, then clearly $D\,(m, \overline{\alpha}\,(n+1))$. If, however, $\neg D\,(m, (\overline{\alpha}n)\,i)$ for some $i \in \{0, 1\}$, then $D\,(m, (\overline{\alpha}n)\,\bar{\imath})$ and $A_i\,(\overline{\alpha}n)$; whence $\alpha\,(n) = \bar{\imath}$ (by the choice of α) and again $D\,(m, \overline{\alpha}\,(n+1))$. $\square$

Corollary 28. WKL *is equivalent to* LLPO+Π^0_1-DC$^\vee$.

A tree S is a *spread* if every element of S has an immediate successor in S: that is,

$$\forall u\ (u \in S \Rightarrow u0 \in S \vee u1 \in S) \tag{6}$$

or, equivalently,

$$\forall u\ ((u \in S \Rightarrow u0 \in S) \vee (u \in S \Rightarrow u1 \in S)) \tag{7}$$

(recall that every tree is assumed to be detachable). By induction every spread is an infinite tree; whence WKL implies a principle that we therefore call the *Weak Spread Lemma* (WSL):

WSL *Every spread has an infinite path.*

For Kleene's time-honoured discovery [17] of an infinite tree without infinite path in recursive mathematics, one cannot expect to prove WKL with constructive means. Its consequence WSL, however, is weak enough to allow for a constructive proof:

Proposition 29. WSL *is provable.*

Proof. Since Δ_0-DC$^\vee$ is provable (Proposition 3), we only need to show that it implies WSL. To this end, let S be a spread. For each $i \in \{0, 1\}$ set

$$A_i\,(u) \equiv (u \in S \Rightarrow ui \in S)\ ,$$

which is a decidable assertion. By (7) we have $\forall u\,(A_0(u) \vee A_1(u))$; whence by Δ_0-DC$^\vee$ there is α with $\forall n\, A_{\alpha(n)}(\overline{\alpha}n)$. Induction on n proves that

$\forall n\ (\overline{\alpha}n \in S)$ for this α. (The case $n = 0$ is $() \in S$; if $\overline{\alpha}n \in S$, then $\overline{\alpha}\,(n+1) \in S$ for $\overline{\alpha}\,(n+1) = (\overline{\alpha}n)\,\alpha(n)$ and $A_{\alpha(n)}(\overline{\alpha}n)$.) □

In this proof we have inferred WSL from Δ_0-DC$^\vee$. Conversely, Δ_0-DC$^\vee$ can be deduced from WSL as follows. Let $A_0\,(u)$ and $A_1\,(u)$ be decidable assertions. Set

$$S = \left\{u : \forall k < |u|\, A_{u(k)}\,(\overline{u}k)\right\} ,$$

which clearly is a tree. If $\forall u\,(A_0(u) \vee A_1(u))$, then (6) holds for this S, because $u \in S$ together with $A_i(u)$ implies $ui \in S$ for $i \in \{0,1\}$. By WSL there is α with $\forall n\ (\overline{\alpha}n \in S)$: that is, $A_{\alpha(k)}(\overline{\alpha}k)$ for every $k < n$ and all n, or simply $\forall n\, A_{\alpha(n)}(\overline{\alpha}n)$.

7. Omniscience principles put in a uniform way

As a complement we rephrase in a uniform way all the omniscience principles but Markov's that have occurred in this paper. To this end we need to fix the *Law of the Excluded Middle* (Γ-LEM) and *De Morgan's Law* (Γ-DML) as restricted to any assertion class Γ:

Γ-LEM $C \vee \neg C$ *for all* $C \in \Gamma$;

Γ-DML $\neg\,(C \wedge D) \Rightarrow \neg C \vee \neg D$ *for all* $C, D \in \Gamma$.

For arbitrary assertion classes Γ, Δ we consider the following principle:

$\boldsymbol{P(\Gamma,\Delta)}$ $(C \Rightarrow D) \Rightarrow \neg C \vee D$ *for all* $C \in \Gamma$ *and* $D \in \Delta$.

We further write Φ for the class of all assertions. The proof of the next lemma is left to the reader as an exercise in intuitionistic propositional logic.

Lemma 30. *Let* Γ *be a class of assertions.*

1. *The following are equivalent:* Γ-LEM*;* $P\,(\Gamma,\Phi)$*;* $P\,(\Phi,\Gamma)$*;* $P\,(\Gamma,\Gamma)$.
2. Γ-DML *is equivalent to* $P\,(\Gamma,\Delta)$ *with* $\Delta = \{\neg C : C \in \Gamma\}$.

Note that LPO, WLPO, LLPO, and WLLPO are nothing but Σ^0_1-LEM, Π^0_1-LEM, Σ^0_1-DML and Π^0_1-DML, respectively. We set

$$\Xi^0_1 = \{\neg B : B \in \Pi^0_1\} = \{\neg\neg A : A \in \Sigma^0_1\}\,.$$

Corollary 31.

1. (a) *The following are equivalent:* LPO*;* $P\left(\Sigma^0_1, \Phi\right)$*;* $P\left(\Phi, \Sigma^0_1\right)$*;* $P\left(\Sigma^0_1, \Sigma^0_1\right)$*.*
   (b) *The following are equivalent:* WLPO*;* $P\left(\Pi^0_1, \Phi\right)$*;* $P\left(\Phi, \Pi^0_1\right)$*;* $P\left(\Pi^0_1, \Pi^0_1\right)$*.*
2. (a) LLPO *is equivalent to* $P\left(\Sigma^0_1, \Pi^0_1\right)$*.*
   (b) WLLPO *is equivalent to* $P\left(\Pi^0_1, \Xi^0_1\right)$*.*

Now let Ψ stand for the class of all negated assertions.

Proposition 32. WLPO*,* $P\left(\Sigma^0_1, \Psi\right)$*, and* $P\left(\Sigma^0_1, \Xi^0_1\right)$ *are equivalent.*

Proof. In view of $\Xi^0_1 \subseteq \Psi$ we only have to check that
(i) WLPO implies $P\left(\Sigma^0_1, \Psi\right)$ and (ii) WLPO follows from $P\left(\Sigma^0_1, \Xi^0_1\right)$. As for (i), let $A \in \Sigma^0_1$ and $F \in \Phi$. Assume that $\neg A \vee \neg\neg A$. Hence if $A \Rightarrow \neg F$ and thus $\neg\neg A \Rightarrow \neg F$, then $\neg A \vee \neg F$. To prove (ii) use $A \Rightarrow \neg\neg A$ for any $A \in \Sigma^0_1$. □

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