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Three Solutions Theorem for a Quasilinear Dirichlet Boundary Value Problem

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Abstract. We consider a Dirichlet boundary value problem driven by the p-Laplacian with the right hand side being a Carathéodory function. The existence of solutions is obtained by the use of a special form of the three critical points theorem.

Keywords: Carathéodory function, Dirichlet problem, p-Laplacian, three critical points theorem, weak solution.

1. Introduction

In this paper we show that the problem

$$\begin{cases} -\Delta_p u & = & \lambda f(x,u) & \text{in } \Omega, \\ u & = & 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

has at least three weak solutions in $W_0^{1,p}(\Omega)$, where $p \in (1,\infty)$ is given and Δ_p stands for the p-Laplacian defined by $\Delta_p u := \operatorname{div}\Big(|\nabla u|_N^{p-2}\nabla u\Big)$. Here $\Omega \subset \mathbb{R}^N$ is a nonempty open and bounded set with the boundary of class C^1, λ is a positive parameter and $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying appropriate growth condition together with its antiderivative given by the definite integral with a variable upper bound.

For the past several years there have been published many papers dealing with the existence of at least three solutions for elliptic problems. As a main tool in proofs the authors were using the three critical points theorem due to Ricceri ([13, Theorem 3.1], [14, Theorem 1]) and its other versions and generalizations (see for example [1, 8, 12, 11, 10]).

Ricceri first applied his theorems to prove the existence of at least three distinct weak solutions in $H_0^1(\Omega)$ of the problem

$$\begin{cases} -\Delta u &= \lambda\big(f(u)+\mu g(u)\big) \quad \text{in } \Omega, \\ u &= 0 \qquad\qquad\qquad\quad\ \text{on } \partial\Omega \end{cases} \tag{2}$$

for all $\lambda \in \Lambda$, where $\Lambda \subset [0,\infty)$ is an open nonempty interval and the functions f, g satisfy appropriate growth conditions together with their antiderivatives. Then he investigated more general both Dirichlet and Neumann problems involving the p-Laplace operator (see [12, 11, 10]). Problem (2) has also been generalized by Goncerz [7] to the p-Laplacian case and the multiplicity result in $W_0^{1,p}(\Omega)$ was obtained.

There were also many other publications in which problems driven by the $p(x)$-Laplacian were discussed. For instance Mihăilescu proved in [9] the existence of at least three weak solutions for a Neumann problem under the assumptions that $\inf_{x\in\overline{\Omega}} p(x) > N \geqslant 3$ and with the right hand side nonlinearity of the form $f(x,t) = |t|^{q(x)-2}t - t$, where $q \in \Big\{h \in C\big(\overline{\Omega}\big) : h(x) > 1 \quad \forall x \in \overline{\Omega}\Big\}$ and $2 < q(x) < \inf_{x\in\overline{\Omega}} p(x)$ for any $x \in \overline{\Omega}$. An analogous result as above but for more general f was established by Wang, Fan and Ge [16].

It is worth to mention that there are papers in which the authors obtained multiplicity results using another methods than three critical points theorem. For example exploiting critical point theory Gasiński and Papageorgiou proved in [6] the existence of five nontrivial solutions (two positive, two nonnegative and the fifth nodal) for a nonlinear Dirichlet elliptic differential equation driven by the p-Laplace operator and with a nonsmooth potential.

Besides the research on the existence and multiplicity of solutions, the problem of the localization of an interval for the parameter λ was considered. For instance in [2] the authors established a theorem which yields the formula for the mentioned interval for an elliptic Dirichlet problem driven by the Laplacian whereas in [3] another formula has been obtained for a non-homogeneous Neumann problem involving the elliptic operator of the form $\operatorname{div}\big(\alpha(|\nabla\cdot|_N)\nabla\cdot\big)$.

The aim of this paper is to generalize the result of Bonanno and Molica Bisci [2] to the general p-Laplace operator. Here any relation between N and p will not be required. The technical approach which has been used in this paper is analogous as in [4, 5].

2. Preliminaries

In this section we recall a three critical points theorem in a convenient form.

Theorem 1. ([2, Theorem 2.1], [3, Theorem 1.2]) *Let X be a reflexive real Banach space, $\Phi\colon X \to \mathbb{R}$ a coercive, continuously Gâteaux differentiable and sequentially* *weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi\colon X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that $\Phi(0) = \Psi(0) = 0$. Assume that there exist $r > 0$ and $\tilde{x} \in X$, with $r < \Phi(\tilde{x})$, such that*

(A1) $\displaystyle \frac{1}{r} \sup_{\Phi(x)\leqslant r} \Psi(x) \;<\; \frac{\Psi(\tilde{x})}{\Phi(\tilde{x})}$,

(A2) *for each* $\lambda \in \Lambda_r := \left(\dfrac{\Phi(\tilde{x})}{\Psi(\tilde{x})},\ \dfrac{r}{\sup_{\Phi(x)\leqslant r} \Psi(x)} \right)$ *the functional* $J_\lambda := \Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$, J_λ has at least three distinct critical points in X.

3. The main result

In this section we formulate and prove the main result concerning the existence of solutions of problem (1) and location of parameter λ. At the beginning we introduce the notations which will be used in the sequel.

Let $N \geqslant N$, $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^N$ be a nonempty open and bounded set with the boundary of class C^1. Denote by $|\cdot|_N$, $\|\cdot\|_\alpha$ the norms in $\mathbb{R}^N$ and $L^\alpha(\Omega)$ respectively, and by c_α the constant of the embedding $W^{1,p}(\Omega) \subset L^\alpha(\Omega)$.

Let X be a Sobolev space $W_0^{1,p}(\Omega)$ with the norm $\|\cdot\| = \left(\displaystyle\int_\Omega |\nabla \cdot|_N^p dx \right)^{\frac{1}{p}}$. The Sobolev critical exponent is defined by

$$p^* := \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ \infty & \text{if } p \geqslant N. \end{cases}$$

By Γ we denote the Gamma function $\Gamma(z) = \displaystyle\int_0^\infty t^{z-1} e^{-t} dt$ for all $z > 0$.

Put $D := \sup\limits_{x\in\Omega} \operatorname{dist}(x, \partial\Omega)$. Then there exists $x_0 \in \Omega$ such that $B(x_0, D) \subset \Omega$. For all $\theta \in (0,1)$ we put $K_\theta := \dfrac{p\Gamma\left(1 + \frac{N}{2}\right)(1-\theta)^p}{\pi^{\frac{N}{2}} D^{N-p}\left(1 - \theta^N\right)}$.

The main result of the paper is following

Theorem 2. *Let $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and denote $F(x,\xi) := \displaystyle\int_0^\xi f(x,t)dt$. Assume that f satisfies the following conditions:*

(H1) $\exists a_1, a_2 \geqslant 0\ \exists q \in \left[1, p^*\right)\ \forall (x,t) \in \Omega \times \mathbb{R} \quad |f(x,t)| \leqslant a_1 + a_2|t|^{q-1}$, (H2) $\exists b > 0\ \exists s \in (0,p)\ \forall \xi \in \mathbb{R} \quad F(x,\xi) \leqslant b\big(1+|\xi|^s\big)$ *for almost all* $x \in \Omega$,

(H3) *Let* $\beta(r) := a_1 c_1 p^{\frac{1}{p}} r^{\frac{1-p}{p}} + \dfrac{a_2}{q} c_q^q p^{\frac{q}{p}} r^{\frac{q-p}{p}}$ *for all* $r > 0$. *There exist* $\delta, \gamma > 0$ *and* $\theta \in (0,1)$ *such that* $\delta^p > K_\theta \gamma^p$ *and* $\dfrac{1}{\delta^p} \inf_{x\in\Omega} F(x,\delta) \ > \ \dfrac{\big(1-\theta^N\big)\beta\big(\gamma^p\big)}{pD^p\theta^N(1-\theta)^p}$.

Then, for each $\lambda \in \Lambda_{(\delta,\gamma)} := \left(\dfrac{\big(1-\theta^N\big)\delta^p}{pD^p\theta^N(1-\theta)^p \inf_{x\in\Omega} F(x,\delta)},\ \dfrac{1}{\beta\big(\gamma^p\big)}\right)$, *problem* (1) *admits at least three distinct weak solutions in* $W_0^{1,p}(\Omega)$.

Proof. For $u \in X$ we define two Gâteaux differentiable functionals

$$\Phi(u) := \frac{1}{p}\|u\|^p, \quad \Psi(u) := \int_\Omega F(x,u)dx.$$

Denote $J_\lambda := \Phi - \lambda\Psi$. We will show that the assumptions of Theorem 1 are fulfilled.

Φ is sequentially weakly lower semicontinuous because it is convex and continuous (see [17, Proposition 41.8]) whereas Ψ' is compact (see [7, Proposition 2.5]).

>From assumption (H1) we have

$$F(x,\xi) \ \leqslant\ a_1|\xi| + \frac{a_2}{q}|\xi|^q \qquad \forall (x,\xi) \in \Omega\times\mathbb{R}$$

and by the use of Sobolev embedding theorem we obtain

$$\Psi(u) \ \leqslant\ a_1c_1\|u\| + \frac{a_2}{q}c_q^q\|u\|^q \qquad \forall u \in X.$$

For $r > 0$ we define $\chi(r) := \dfrac{1}{r} \sup\limits_{u\in\Phi^{-1}((-\infty,r])} \Psi(u)$. For every $u \in X$ such that $\Phi(u) \leqslant r$ the following inequality holds:

$$\Psi(u) \ \leqslant\ a_1c_1p^{\frac{1}{p}}r^{\frac{1}{p}} + \frac{a_2}{q}c_q^q p^{\frac{q}{p}} r^{\frac{q}{p}}.$$

So, in particular,

$$\sup_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) \ \leqslant\ a_1c_1p^{\frac{1}{p}}r^{\frac{1}{p}} + \frac{a_2}{q}c_q^q p^{\frac{q}{p}} r^{\frac{q}{p}}$$

and hence

$$\chi(r) \ \leqslant\ a_1c_1p^{\frac{1}{p}}r^{\frac{1-p}{p}} + \frac{a_2}{q}c_q^q p^{\frac{q}{p}} r^{\frac{q-p}{p}} \ = \ \beta(r) \qquad \forall r > 0. \tag{3}$$

Put

$$u_\delta(x) = \begin{cases} 0 & \text{if } x \in \Omega\setminus B(x_0,D),\\ \frac{\delta}{D-\theta D}\big(D-|x-x_0|\big) & \text{if } x \in B(x_0,D)\setminus B(x_0,\theta D),\\ \delta & \text{if } x \in B(x_0,\theta D), \end{cases}$$

where δ and θ are as in (H3). It is easy to check that $u_\delta \in X$. Then

$$\Phi(u_\delta) \;=\; \frac{\delta^p \pi^{\frac{N}{2}} D^{N-p}\big(1-\theta^N\big)}{p\Gamma\big(1+\frac{N}{2}\big)(1-\theta)^p} \;=\; \frac{\delta^p}{K_\theta} \;>\; \gamma^p.$$

>From the definition of u_δ and (H3), we get

$$\Psi(u_\delta) \;\geqslant\; \int_{B(x_0,\theta D)} F(x,\delta)dx \;\geqslant\; \frac{\pi^{\frac{N}{2}}(\theta D)^N}{\Gamma\big(1+\frac{N}{2}\big)} \inf_{x\in\Omega} F(x,\delta) \;>\; 0$$

and from (3) and (H3) again, we obtain

$$\begin{aligned}\frac{\Psi(u_\delta)}{\Phi(u_\delta)} \;&\geqslant\; \frac{p\Gamma\big(1+\frac{N}{2}\big)(1-\theta)^p}{\delta^p\pi^{\frac{N}{2}}D^{N-p}\big(1-\theta^N\big)} \cdot \frac{\pi^{\frac{N}{2}}(\theta D)^N}{\Gamma\big(1+\frac{N}{2}\big)} \inf_{x\in\Omega} F(x,\delta) \\ &=\; \frac{pD^p\theta^N(1-\theta)^p}{\delta^p\big(1-\theta^N\big)} \inf_{x\in\Omega} F(x,\delta) \;>\; \beta\big(\gamma^p\big) \;\geqslant\; \chi\big(\gamma^p\big).\end{aligned}$$

We have shown that assumption (A1) of Theorem 1 is satisfied with $r=\gamma^p$ and $\tilde{x}=u_\delta$.

If $u\in X$, then $|u|^s \in L^{\frac{p}{s}}(\Omega)$ and by the Hölder inequality, we have

$$\int_\Omega |u|^s dx \;\leqslant\; \Big(\int_\Omega \big(|u|^s\big)^{\frac{p}{s}} dx\Big)^{\frac{s}{p}} \Big(\int_\Omega 1^{\frac{p}{p-s}} dx\Big)^{\frac{p-s}{p}} \;\leqslant\; \big(\mathrm{meas}(\Omega)\big)^{\frac{p-s}{p}} c_p^s \|u\|^s \qquad \forall u\in X.$$

>From this and (H2), it follows that for every $\lambda>0$

$$\begin{aligned} J_\lambda(u) \;&=\; \Phi(u)-\lambda\Psi(u) \;\geqslant\; \frac{1}{p}\|u\|^p - b\lambda\int_\Omega \big(1+|u|^s\big)dx \\ &\geqslant\; \frac{1}{p}\|u\|^p - b\lambda\mathrm{meas}(\Omega) - b\lambda\big(\mathrm{meas}(\Omega)\big)^{\frac{p-s}{p}} c_p^s\|u\|^s \;\xrightarrow{\|u\|\to\infty}\; \infty,\end{aligned}$$

so, in particular, J_λ is coercive for every $\lambda \in \Lambda_{(\delta,\gamma)} = \left(\dfrac{\Phi(u_\delta)}{\Psi(u_\delta)},\ \dfrac{\gamma^p}{\sup_{\Phi(u)\leqslant\gamma^p}\Psi(u)}\right)$.

We have shown that assumption (A2) of Theorem 1 is also satisfied. As a result the functional J_λ has at least three distinct critical points in X which are weak solutions of problem (1). □

4. Examples

Here we present examples of problems to which one can use Theorem 2.

Example 3. Let $N \in \mathbb{N}$, $p \in (1, \infty)$ and $\Omega \subset \mathbb{R}^N$ be a nonempty open and bounded set with the boundary of class C^1. Fix $q \in \big(p, p^*\big)$, $s \in (0, p)$ and put

$$\widetilde{K}_\theta := \max\left\{1,\ K_\theta^{\frac{1}{p}},\ \left(\frac{q\big(1-\theta^N\big)}{pD^p\theta^N(1-\theta)^p}\Big(c_1 q p^{\frac{1}{p}} + c_q^q p^{\frac{q}{p}}\Big)\right)^{\frac{1}{q-p}}\right\}, \tag{4}$$

for some $\theta \in (0, 1)$.

Let $\delta > \widetilde{K}_\theta$ and $g\colon \Omega \to \mathbb{R}$ be a measurable function such that $g(x) \in [0, 1]$ for all $x \in \Omega$, and define Carathéodory function $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$f(x,t) = \begin{cases} g(x) + |t|^{q-1} & \text{if } t \leqslant \delta, \\ g(x) + \delta^{q-s}|t|^{s-1} & \text{if } t > \delta. \end{cases}$$

Obviously, $|f(x,t)| \leqslant 1 + |t|^{q-1}$ for all $(x,t) \in \Omega \times \mathbb{R}$ so we can take $a_1 = a_2 = 1$. It is easy to show that for almost every $x \in \Omega$, we have

$$F(x,\xi) = \begin{cases} g(x)\xi - \dfrac{(-\xi)^q}{q} & \text{for all } \xi < 0, \\[2ex] g(x)\xi + \dfrac{\xi^q}{q} & \text{for all } \xi \in [0,\delta], \\[2ex] g(x)\xi + \dfrac{\delta^q}{q} + \dfrac{\delta^{q-s}\xi^s}{s} - \dfrac{\delta^q}{s} & \text{for all } \xi > \delta, \end{cases}$$

and hence

$$F(x,\xi) \leqslant \left(\delta + \frac{\delta^q}{s}\right)\Big(1 + |\xi|^{\max\{1,s\}}\Big)$$

for all $\xi \in \mathbb{R}$ and almost all $x \in \Omega$. We can choose $b = \delta + \dfrac{\delta^q}{s}$.

If we fix $\gamma = 1$, then we have $\delta^p > \widetilde{K}_\theta^p > K_\theta \gamma^p$ and moreover

$$\begin{aligned} \frac{1}{\delta^p}\inf_{x\in\Omega} F(x,\delta) &= \frac{1}{\delta^p}\left(\frac{\delta^q}{q} + \delta \inf_{x\in\Omega} g(x)\right) \geqslant \frac{\delta^{q-p}}{q} \\ &\geqslant \frac{1}{q}\cdot\frac{q(1-\theta)^N}{pD^p\theta^N(1-\theta)^p}\Big(c_1 q p^{\frac{1}{p}} + c_q^q p^{\frac{q}{p}}\Big) \\ &= \frac{(1-\theta)^N\beta(1)}{pD^p\theta^N(1-\theta)^p}. \end{aligned}$$

We get that for each $\lambda \in \left(\dfrac{\big(1-\theta^N\big)\delta^p}{pD^p\theta^N(1-\theta)^p \inf_{x\in\Omega} F(x,\delta)},\ \dfrac{1}{\beta(1)}\right)$ problem (1) has at least three distinct weak solutions in $W_0^{1,p}(\Omega)$.

For the convenience of the reader we recall some facts before moving to the next example.

Theorem 4. (Sobolev–Gagliardo–Nirenberg) *If $p \in [1, N)$, then there exists a constant $c = c(N,p) > 0$ such that*

$$\|u\|_{L^{p^*}(\mathbb{R}^N)} \leqslant c\|u\|_{W_0^{1,p}(\mathbb{R}^N)} \tag{5}$$

for all $u \in L^{1,p}(\mathbb{R}^N) = \Big\{u \in L^p_{\mathrm{loc}}(\mathbb{R}^N) : \nabla u \in L^p\big(\mathbb{R}^N;\mathbb{R}^N\big)\Big\}$ decaying at infinity. In particular, the embedding $W^{1,p}(\mathbb{R}^N) \subset L^q(\mathbb{R}^N)$ is continuous for each $q \in [p, p^*]$.

If $p \in (1, N)$, then

$$c = \pi^{-\frac{1}{2}} N^{-\frac{1}{p}} \left(\frac{p-1}{N-p}\right)^{1-\frac{1}{p}} \left(\frac{\Gamma\big(1+\frac{N}{2}\big)\Gamma(N)}{\Gamma\Big(\frac{N}{p}\Big)\Gamma\Big(1+N-\frac{N}{p}\Big)}\right)^{\frac{1}{N}}$$

is the best constant in inequality (5) (see [15]).

Proposition 5. *If $u \in L^p(\mathbb{R})$ for some $p \in [1,\infty)$, then $\liminf\limits_{x\to\pm\infty} |v(x)| = 0$ for all v from the equivalence class of u, and "$\liminf$" cannot be replaced by "$\lim$".*

Proposition 6. *Let $1 \leqslant \alpha_1 < \alpha_2 \leqslant \infty$ and $U \subset \mathbb{R}^N$ be a bounded set. Then*

$$\|u\|_{L^{\alpha_1}(U)} \leqslant \big(\mathrm{meas}(U)\big)^{\frac{1}{\alpha_1}-\frac{1}{\alpha_2}} \|u\|_{L^{\alpha_2}(U)} \qquad \forall u \in L^{\alpha_2}(U).$$

Now we can formulate the next example.

Example 7. Taking into account Example 3, let $N = 6$ and $p = 3$. Then $p^* = 6$. We take $q = 4 \in (p, p^*)$ and $s = \dfrac{5}{2} \in (0,p)$.

Let $\Omega = B(0,1) \subset \mathbb{R}^6$. Then $D = 1$, $\mathrm{meas}(\Omega) = \dfrac{\pi^3}{6}$ and since $\theta \in (0,1)$, we obtain

$$K_\theta = \frac{18(1-\theta)^3}{\pi^3\big(1-\theta^6\big)} < \frac{18}{\pi^3} < 1.$$

After easy calculations we have

$$c = \sqrt{\frac{2}{\pi}} \sqrt[6]{\frac{5}{3^5}} \approx 0.4177$$

and using Proposition 6 and Theorem 4, we obtain

$$c_1 \leqslant \left(\frac{\pi^3}{6}\right)^{1-\frac{1}{6}} \sqrt{\frac{2}{\pi}} \sqrt[6]{\frac{5}{3^5}} = \frac{\pi^2 \sqrt[6]{5}}{\sqrt[3]{2\cdot 3^5}} \approx 1.6415$$

and

$$c_4 \leqslant \left(\frac{\pi^3}{6}\right)^{\frac{1}{4}-\frac{1}{6}} \sqrt{\frac{2}{\pi}} \sqrt[6]{\frac{5}{3^5}} = \frac{\sqrt[6]{5}}{\sqrt[4]{\pi}} \sqrt[12]{\frac{2^5}{3^{11}}} \approx 0.4789.$$

The function $y(\theta) = \dfrac{(1-\theta^6)}{\theta^6(1-\theta)^3}$ attains its minimal value at $\theta_0 \approx 0.692$ and $y(\theta_0) \approx 277.456$, so we will obtain the biggest interval for parameter λ by taking $\theta = \theta_0$. Hence

$$\begin{aligned}
\left(\frac{q\big(1-\theta_0^N\big)}{pD^p\theta_0^N(1-\theta_0)^p}\Big(c_1 q p^{\frac{1}{p}} + c_q^q p^{\frac{q}{p}}\Big)\right)^{\frac{1}{q-p}} &= \frac{4\big(1-\theta_0^6\big)}{3\theta_0^6(1-\theta_0)^3}\Big(4\cdot 3^{\frac{1}{3}}c_1 + 3^{\frac{4}{3}}c_4^4\Big)\\
&\leqslant \frac{4\big(1-\theta_0^6\big)}{3\theta_0^6(1-\theta_0)^3}\left(4\cdot 3^{\frac{1}{3}}\cdot\frac{5^{\frac{1}{6}}\pi^2}{2^{\frac{1}{3}}\cdot 3^{\frac{5}{3}}} + 3^{\frac{4}{3}}\cdot\frac{2^{\frac{5}{3}}\cdot 5^{\frac{2}{3}}}{3^{\frac{11}{3}}\pi}\right)\\
&= \frac{1-\theta_0^6}{\theta_0^6(1-\theta_0)^3}\left(\frac{2^{\frac{11}{3}}\cdot 5^{\frac{1}{6}}\pi^2}{3^{\frac{7}{3}}} + \frac{2^{\frac{11}{3}}\cdot 5^{\frac{2}{3}}}{3^{\frac{10}{3}}\pi}\right) \approx 277.456\cdot 12.9302\\
&\approx 3587.5616 < 3600.
\end{aligned}$$

Fix $\gamma = 1$ and $\delta = 3600 > \widetilde{K}_{\theta_0} > K_{\theta_0}^{\frac{1}{p}}\gamma$. Define $g\colon \Omega \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{for } x \in \big\{z \in B(0,1):\ z_6 \geqslant 0\big\},\\ 0 & \text{for } x \in \big\{z \in B(0,1):\ z_6 < 0\big\}\end{cases}$$

and a Carathéodory function $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ by

$$f(x,t) = \begin{cases} g(x) + |t|^3 & \text{if } t \leqslant 3600,\\ g(x) + 216000t^{\frac{3}{2}} & \text{if } t > 3600.\end{cases}$$

We take $a_1 = a_2 = 1$ as we have shown in Example 3. Then

$$\begin{aligned}
\beta(1) &= a_1 c_1 p^{\frac{1}{p}} + \frac{a_2}{q}c_q^q p^{\frac{q}{p}} = 3^{\frac{1}{3}}c_1 + \frac{3^{\frac{4}{3}}}{4}c_4^4 \leqslant 3^{\frac{1}{3}}\cdot\frac{5^{\frac{1}{6}}\pi^2}{2^{\frac{1}{3}}\cdot 3^{\frac{5}{3}}} + \frac{3^{\frac{4}{3}}}{4}\cdot\frac{2^{\frac{5}{3}}\cdot 5^{\frac{2}{3}}}{3^{\frac{11}{3}}\pi}\\
&= \frac{5^{\frac{1}{6}}\pi^2}{2^{\frac{1}{3}}\cdot 3^{\frac{4}{3}}} + \frac{5^{\frac{2}{3}}}{2^{\frac{1}{3}}\cdot 3^{\frac{7}{3}}\pi} < 2.4245
\end{aligned}$$

and hence

$$\frac{1}{\beta(1)} > \frac{1}{2.4245} > 0.4124.$$

We have also shown in Example 3 that

$$\inf_{x\in\Omega} F(x,\delta) = \frac{\delta^q}{q} + \delta\inf_{x\in\Omega} g(x)$$

and therefore

$$\inf_{x\in\Omega} F(x,3600) = \frac{3600^4}{4} + 3600\inf_{x\in\Omega} g(x) = \frac{3600^4}{4}.$$

Eventually we obtain

$$\frac{\big(1-\theta_0^N\big)\delta^p}{pD^p\theta_0^N(1-\theta_0)^p\inf_{x\in\Omega}F(x,\delta)} = \frac{1-\theta_0^6}{\theta_0^6(1-\theta_0)^3}\cdot\frac{4\cdot 3600^3}{3\cdot 3600^4} \approx 277.456\cdot\frac{1}{2700} < 0.1028.$$

In particular, for each $\lambda \in \big(0.1028,\ 0.4124\big) \subset \Lambda_{(3600,1)}$ the problem

$$\begin{cases} -\mathrm{div}\big(|\nabla u|\nabla u\big) &= \lambda f(x,u) & \text{in } B(0,1),\\ u &= 0 & \text{on } S(0,1)\end{cases}$$

admits at least three distinct weak solutions in $W_0^{1,3}\big(B(0,1)\big)$.

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