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Quasidifferentiable Calculus and Minimal Pairs of Compact Convex Sets

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Abstract. The quasidifferential calculus developed by V.F. Demyanov and A.M. Rubinov provides a complete analogon to the classical calculus of differentiation for a wide class of nonsmooth functions. Although this looks at the first glance as a generalized subgradient calculus for pairs of subdifferentials it turns out that, after a more detailed analysis, the quasidifferential calculus is a kind of Fréchet-differentiation whose gradients are elements of a suitable Minkowski–Rådström–Hörmander space. One aim of the paper is to point out this fact. The main results in this direction are Theorem 1 and Theorem 5. Since the elements of the Minkowski–Rådström–Hörmander space are not uniquely determined, we focus our attention in the second part of the paper to smallest possible representations of quasidifferentials, i.e. to minimal representations. Here the main results are two necessary minimality criteria, which are stated in Theorem 9 and Theorem 11.

Keywords: nonsmooth optimization, generalized convexity.

1. Introduction and notations

This is a survey paper on common work with Jerzy Grzybowski from Poznań.

Pairs of compact convex sets arise in the quasidifferential calculus of V.F. Demyanov and A.M. Rubinov as sub- and superdifferentials of quasidifferentiable functions (see [3]). The general framework for the investigation of pairs of nonempty compact convex sets is the Minkowski–Rådström–Hörmander space (see [9] and [14]). Since this space is inherently infinite dimensional, we will state our results in the terminology of topological vector spaces.

The notation *topological vector space* is a collective term for a large class of not necessarily finite dimensional vector spaces endowed with a Hausdorff topology, such that the vector addition and the multiplication by scalars is continuous. It includes Banach- and Hilbert-spaces, as well as all different types of locally convex spaces and also nonlocally convex spaces, such as the space of measurable functions, endowed with the topology of convergence in measure. In the finite dimensional case everything reduces to the Euclidean space with the standard topology (see [11]).

Therefore we will use throughout this paper the following notation: Let $X = (X,\tau)$ be a real topological vector space and $\mathcal{B}(X)$ (resp. $\mathcal{K}(X)$) the set of all nonempty bounded closed (resp. compact) convex subsets of X. For nonempty $A,B \subseteq X: \quad A+B=\{x=a+b \mid a\in A \text{ and } b\in B\}$ denotes the *algebraic sum* and we denote by $A \dotplus B = \operatorname{cl}(\{x=a+b \mid a\in A \text{ and } b\in B\})$ the *Minkowski sum* which is the closure of $A+B$. We write $\operatorname{cl}(A)=\bar{A}$ for the closure of $A\subset X$ with respect to the topology τ. For compact convex sets this coincides with the usual definition of the Minkowski sum since for $A,B\in\mathcal{K}(X): \quad A\dotplus B = A+B$ holds. Since $\mathcal{B}(X)$ satisfies the order cancellation law, i.e. for $A,B,C\in\mathcal{B}(X)$ the inclusion $A\dotplus B\subseteq B\dotplus C$ implies $A\subseteq C$, (see [19] and [14], Theorem 3.2.1) the sets $\mathcal{B}(X)$ and $\mathcal{K}(X)$ endowed with the Minkowski sum are commutative semigroups with the cancellation property. A set $A\in\mathcal{B}(X)$ is called a *summand* of $C\in\mathcal{B}(X)$ if there exists a set $B\in\mathcal{B}(X)$ with $A\dotplus B=C$.

Let us fix some further notations: Let $f\in X'$ be a continuous linear functional. Then we denote for $A\in\mathcal{K}(X)$ by $H_f(A)=\{z\in A \mid f(z)=\max_{y\in A} f(y)\}$ the (maximal) *face* of A with respect to f. For $A,B\in\mathcal{K}(X)$ and $f\in X'$ holds the following identity: $H_f(A+B)=H_f(A)+H_f(B)$.

For $A\in\mathcal{B}(X)$ we denote by $\mathcal{E}(A)$ the set of extremal points of A and by $\mathcal{E}_0(A)$ the set of its exposed points. Recall that $x_0\in A$ is an *exposed point* if and only if there exists an $f\in X'\setminus\{0\}$ such that $H_f(A)=\{x_0\}$.

For two sets $A,B\in\mathcal{B}(X)$ we will use the notation $A\vee B=\operatorname{cl}\operatorname{conv}(A\cup B)$. A.G. Pinsker [16] proved the following identity for $A,B,C\in\mathcal{B}(X)$:

$$(A\dotplus C)\vee(B\dotplus C)=C\dotplus(A\vee B).$$

We will use the abbreviation $A\dotplus B\vee C$ for $A\dotplus(B\vee C)$ and $C\dotplus d$ for $C\dotplus\{d\}$.

An equivalence relation on $\mathcal{B}^2(X)=\mathcal{B}(X)\times\mathcal{B}(X)$ is given by $(A,B)\sim(C,D)$ if and only if $A\dotplus D=B\dotplus C$ and an ordering by the relation: $(A,B)\leq(C,D)$ if and only if $A\subseteq C$ and $B\subseteq D$. The equivalence class of (A,B) is denoted by $[A,B]$.

We call a pair $(A,B)\in\mathcal{B}^2(X)$ *minimal* if it is minimal in the class $[A,B]$, i.e. if for any pair $(C,D)\in[A,B]$ the relation $(C,D)\leq(A,B)$ implies that $(C,D)=(A,B)$.

Minimal pairs of nonempty bounded closed convex sets have many interesting properties which where studied in a series of papers (see for instance [14] and the references cited therein). In particular, it follows from the Lemma of Kuratowski–Zorn that every equivalence class in $\mathcal{K}^2(X)$ has a minimal element. This is not longer true for $\mathcal{B}^2(X)$ (see [8]).

In the 2-dimensional case, equivalent minimal pairs of compact convex sets are uniquely determined up to translations, which is not longer true for the 3-dimensional case. J. Grzybowski and R. Urbański [8] showed under the assumption of the continuum hypothesis, that if there exist any two equivalent minimal pairs of compact convex sets which are not related by a translation, then there exists already a continuous family of equivalent minimal pairs which are also not related by translations.

A pair $(A, B) \in \mathcal{B}^2(X)$ is called *convex* if $A \cup B$ is convex. It follows from the order cancellation law that in this case every pair $(C, D) \in [A, B]$ is also convex, so that the whole class can be considered as convex (see [14]).

2. Minkowski–Rådström–Hörmander space

In 1954 L. Hörmander [9] investigated the equivalence classes of pairs of nonempty compact convex sets for a locally convex space in terms of their support functions. In particular he proved, that for a topological vector space (X, τ) an arbitrary sublinear function $p : X \to \mathbb{R}$ is continuous in the topology τ if and only if its subdifferential at the origin $\partial p|_0 = \{v \in X^{'} \mid \langle v, x\rangle \leq p(x),\ x \in X\}$ is an element of $\mathcal{K}(X^{'})$ of all nonempty compact convex subsets in the weak-$*$-topology $\sigma(X, X^{'})$ of X' and that $p : X \to \mathbb{R}$ has the representation $p(x) = p_A(x) = \max_{a\in A} \langle a, x\rangle$ with $A = \partial p|_0 \in \mathcal{K}(X^{'})$. Hence the sublinear functions are exactly the support functions.

Now we denote for a topological vector space (X, τ) by $\mathcal{P}(X)$ the set of all continuous sublinear functions defined on X and by $\mathrm{DCH}(X) = \{\varphi = p - q \mid p, q \in \mathcal{P}(X)\}$ the real vector space of all differences of continuous sublinear functions (see [14]). The notation DCH stand for *difference of convex homogeneous* because *sublinear* is equivalent to *convex homogeneous.* In other words, the elements of the space $\mathrm{DCH}(X)$ are the pointwise differences of support functions.

With respect to the pointwise ordering of functions given by $\varphi \leq \psi$ if and only if $\varphi(x) \leq \psi(x)$ holds for every $x \in X$, the space $(\mathrm{DCH}(X), \leq)$ is a vector lattice.

Let us now assign to $\varphi \in \mathrm{DCH}(X)$ the set $[\varphi] = \{(\partial\, p|_0, \partial q|_0) \mid$ with $\varphi = p - q,\ p, q \in \mathcal{P}(X)\}$. To formalize this assignment more precisely, let us consider the set $\mathcal{K}^2(X')/_\sim$ of all equivalence class of pairs $(A, B) \in \mathcal{K}^2(X')$. In 1966 A.G. Pinsker [16] introduced the following ordering on $\mathcal{K}^2(X')/_\sim$, namely: $[A, B] \preceq [C, D] \iff A + D \subseteq B + C$, which is independent of the special choice of representatives, because of the order cancellation law.

The space $\left(\mathcal{K}^2(X')/_\sim, \preceq\right)$ is called the Minkowski–Rådström–Hörmander space of classes of pairs of nonempty compact convex sets (see [14]). It is a complete vector lattice and a direct calculation shows that the assignment:

$$\mathrm{DCH}(X) \longrightarrow \mathcal{K}^2(X')/_\sim \ \text{ with }\ \varphi \mapsto [\varphi] = \{(\partial p|_0, \partial q|_0) \mid \text{ with } \varphi = p - q,\ p, q \in \mathcal{P}(X)\}$$

is a lattice isomorphism, called *Minkowski duality* (see [14], Theorem 3.4.3).

The following result is taken from [13], Theorem 8.1.26.

Theorem 1 *Let $(X, \|\cdot\|)$ be a Banach space. Then the space*

$$\mathrm{DCH}(X) = \{\varphi = p - q \mid p, q \text{ are sublinear and continuous}\}$$

endowed with the norm $\|\cdot\|_\Delta$ given by

$$\|\varphi\|_\Delta = \inf_{\substack{p,q \\ \varphi = p-q}} \left\{ \max \left\{ \sup_{\|x\|\le 1} p(x), \sup_{\|x\|\le 1} q(x) \right\} \right\},$$

where the infimum is taken over all continuous sublinear functions p, q such that $\varphi = p - q$, is a Banach space.

Proof: First we show that $\|\varphi\|_\Delta$ is well defined for $\varphi \in \mathrm{DCH}(X)$ and that $\|\cdot\|_\Delta$ is a norm.

Let $\varphi = p - q \in \mathrm{DCH}(X)$ be given. From $p(0) = q(0) = 0$ it follows that $\|\varphi\|_\Delta \ge 0$. Since p, q are Lipschitz continuous, $\|\varphi\|_\Delta < +\infty$. From the definition of $\|\cdot\|_\Delta$ it follows that for all $\varphi \in \mathrm{DCH}(X)$ and $t \in \mathbb{R}$ the homogeneity condition $\|t\varphi\|_\Delta = |t|\|\varphi\|_\Delta$ holds. Next we prove the triangle inequality:

Let $\varepsilon > 0$ be given and let p_1, p_2, q_1, q_2 be continuous sublinear functions with $\varphi_1 = p_1 - q_1,\ \varphi_2 = p_2 - q_2 \in \mathrm{DCH}(X)$ and

$$\|\varphi_1\|_\Delta \le \max \left\{ \sup_{\|x\|\le 1} p_1(x), \sup_{\|x\|\le 1} q_1(x) \right\} \le \|\varphi_1\|_\Delta + \varepsilon,$$

$$\|\varphi_2\|_\Delta \le \max \left\{ \sup_{\|x\|\le 1} p_2(x), \sup_{\|x\|\le 1} q_2(x) \right\} \le \|\varphi_2\|_\Delta + \varepsilon.$$

Now we have

$$\begin{aligned}
\|\varphi_1 + \varphi_2\|_\Delta &\le \max \left\{ \sup_{\|x\|\le 1} [p_1(x) + p_2(x)], \sup_{\|x\|\le 1} [q_1(x) + q_2(x)] \right\} \\
&\le \max \left\{ \sup_{\|x\|\le 1} p_1(x) + \sup_{\|x\|\le 1} p_2(x), \sup_{\|x\|\le 1} q_1(x) + \sup_{\|x\|\le 1} q_2(x) \right\} \\
&\le \max \left\{ \sup_{\|x\|\le 1} p_1(x), \sup_{\|x\|\le 1} q_1(x) \right\} + \max \left\{ \sup_{\|x\|\le 1} p_2(x), \sup_{\|x\|\le 1} q_2(x) \right\} \\
&\le \|\varphi_1\|_\Delta + \|\varphi_2\|_\Delta + 2\varepsilon.
\end{aligned}$$

The triangle inequality follows from the arbitrariness of $\varepsilon > 0$. In this way we have proved that $\mathrm{DCH}(X)$ is a normed vector space.

Now we show that the space $\mathrm{DCH}(X)$ is complete. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathrm{DCH}(X)$ such that

$$\|\varphi_n\|_\Delta < \frac{1}{2^n}.$$

We show that the series $\sum_{n=1}^{\infty} \varphi_n$ is convergent in the space $\mathrm{DCH}(X)$. This convergence implies obviously the completeness of the space $\mathrm{DCH}(X)$.

Let ε be an arbitrary positive number and let $p_n^\varepsilon, q_n^\varepsilon$ be continuous sublinear functions with $\varphi_n = p_n^\varepsilon - q_n^\varepsilon$ and

$$\|\varphi_n\|_\Delta \leq \max \left\{ \sup_{\|x\|\leq 1} p_n^\varepsilon(x), \sup_{\|x\|\leq 1} q_n^\varepsilon(x) \right\} \leq \|\varphi_n\|_\Delta + \frac{\varepsilon}{2^n}.$$

It follows from the definition of the norm $\|\cdot\|_\Delta$ that the series $\sum_{n=1}^\infty p_n^\varepsilon$ and $\sum_{n=1}^\infty q_n^\varepsilon$ are uniformly convergent on the unit ball of X, thus they have limits $p^\varepsilon,\ q^\varepsilon$ which are continuous sublinear functions.

Moreover, the sequences $\left(p^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$ and $\left(q^{\frac{1}{n}}\right)_{n\in\mathbb{N}}$ have limits p and q which are also continuous sublinear functions. We write $\psi(x) = p(x) - q(x)$. Clearly, $\psi \in \mathrm{DCH}(X)$. It remains to show that ψ is a limit of the series $\sum_{n=1}^\infty \varphi_n$.

From the above assumptions and a simple calculation it follows that there exists an index m_ε such that for all $m \geq m_\varepsilon$

$$\left\|\sum_{n=1}^m \varphi_n - (p^\varepsilon - q^\varepsilon)\right\|_\Delta \leq 2\varepsilon.$$

Since $p^{\frac{1}{n}},\ q^{\frac{1}{n}}$ have the limits $p,\ q$, there exists an $\widetilde{m}_\varepsilon$ such that for $m \geq \widetilde{m}_\varepsilon$

$$\left\|\sum_{n=1}^m \varphi_n - (p - q)\right\|_\Delta \leq 2\varepsilon.$$

The arbitrariness of $\varepsilon > 0$ implies that the series $\sum_{n=1}^\infty \varphi_n$ is norm-convergent in the space $\mathrm{DCH}(X)$. □

3. Quasidifferentiable functions

We give a short survey about the quasidifferential calculus which was introduced by V.F. Demyanov and A.M. Rubinov [3]. We begin with max–min combinations of smooth functions:

3.1. Max–min combinations of smooth functions

Let $U \subseteq \mathbb{R}^n$ be an open subset and let $f, f_1, ..., f_m : U \longrightarrow \mathbb{R}$ be continuous functions. If $I(x) = \{i \in \{1, ..., m\} | f_i(x) = f(x)\}$ is nonempty at every point $x \in U$, then f is called a *continuous selection* of the functions $f_1, ..., f_m$. By $CS(f_1, \ldots, f_m)$ the set of all continuous selections of $f_1, \ldots, f_m$ is denoted. A continuous selection of differentiable functions is called a *piecewise differentiable function.* A typical example of a continuous selection is $f(x) = \max_{i\in I} \min_{j\in M_i} f_j(x)$, where I is a finite index set and $M_i \subseteq \{1,\dots,m\}$.

Note that every continuous selection f of C^1-functions $f_1,\dots,f_m : \mathbb{R}^n \longrightarrow \mathbb{R}$ is locally Lipschitz continuous and that for the Clarke subdifferential holds:

$$\partial f(x) = \operatorname{conv}\{\nabla f_i(x) \mid i \in \hat{I}(x)\}$$

with

$$\hat{I}(x) = \{i \in I(x) \mid x \in \operatorname{cl}\operatorname{int}\{z \in \mathbb{R}^n \mid f(z) = f_i(z)\}\}$$

(see [1]).

$I(x)$ is called the *active index set* and $\hat{I}(x)$ is called the *essential active index set.*

Proposition 2 *Let $U \subseteq \mathbb{R}^n$ be an open set and $f_1,...,f_m : U \longrightarrow \mathbb{R}$ be C^1-functions. Then the max-min combination*

$$f(x) = \max_{i\in\{1,\dots,r\}} \min_{j\in M_i} f_j$$

is directionally differentiable at every point $x_0 \in U$ and for the directional derivative of f in the direction $v \in \mathbb{R}^n$ holds:

$$\begin{aligned} v \quad \mapsto \quad & df(x_0,v) = \left.\frac{df}{dv}\right|_{x_0} = \lim_{\substack{\alpha\to 0\\ \alpha>0}} \frac{f(x_0+\alpha v) - f(x_0)}{\alpha} \\ = \quad & \max_{i\in\{1,\dots,r\}} \min_{j\in I_i(x_0)} \langle \nabla f_j(x_0), v\rangle \\ = \quad & \max_{i\in\{1,\dots,r\}} \{ \sum_{\substack{k\in\{1,\dots,r\}\\ k\neq i}} [\max_{j\in I_k(x_0)} \langle -\nabla f_j(x_0), v\rangle]\} \\ & - \sum_{k\in\{1,\dots,r\}} [\max_{j\in I_k(x_0)} \langle -\nabla f_j(x_0, v\rangle] \end{aligned}$$

with $I_k(x_0) = M_k \cap \hat{I}(x_0)$.

Proof: Let $U \subseteq \mathbb{R}^n$ be an open set and $f_1,...,f_m : U \longrightarrow \mathbb{R}$ be C^1-functions. First we note: Let $\varphi_1,\varphi_2 : [t_0,t_1] \longrightarrow \mathbb{R}$ be C^1-functions in one variable with $\varphi_1(t_0) = \varphi_2(t_0)$ and $\varphi_1(t) \le \varphi_2(t)$ for $t \in V$, where V is some (righthand) neighborhood of t_0. Then for the righthand side directional derivatives holds: $\varphi_1'^{+}(t_0) \le \varphi_2'^{+}(t_0)$.

Now we consider the function $\max\{f,g\}$. When $f(x_0) \neq g(x_0)$, the situation is clear, since for α small enough $\max\{f(x_0+\alpha h), g(x_0+\alpha h)\}$ is equal either $f(x_0+\alpha h)$ or $g(x_0+\alpha h)$ and one has only to consider one of the functions f or g. Now assume that $f(x_0) = g(x_0)$. Then we have by the above observation:

$$\left.\frac{d(\max\{f,g\})}{dh}\right|_{x_0} = \lim_{\alpha\downarrow 0} \frac{\max\{f(x_0+\alpha h), g(x_0+\alpha h)\} - f(x_0)}{\alpha}$$

$$= \max\left\{\lim_{\alpha\downarrow 0} \frac{f(x_0+\alpha h) - f(x_0)}{\alpha},\ \lim_{\alpha\downarrow 0} \frac{g(x_0+\alpha h) - g(x_0)}{\alpha}\right\} = \max\ \left\{ \left.\frac{df}{dh}\right|_{x_0}, \left.\frac{dg}{dh}\right|_{x_0} \right\}.$$

For the minimum we use the formula $\min\{f(x), g(x)\} = -\max\{-f(x), -g(x)\}$ and this gives that

$$\left.\frac{d}{dv}\right|_{x_0}\left(\max_{i\in\{1,\dots,r\}}\min_{j\in M_i} f_j\right) = \max_{i\in\{1,\dots,r\}}\min_{j\in I_i(x_0)}\langle\nabla f_j(x_0), v\rangle$$

with $I_k(x_0) = M_k \cap \hat{I}(x_0)$.

The second part of the assertion follows from the following identity for max–min combinations of linear functions, which is proved in [3] (see also [14], Formula 10.1.1)

$$\max_{i\in\{1,\dots,r\}}\min_{j\in M_i}\langle a_j, x\rangle = \max_{i\in\{1,\dots,r\}}\left\{\sum_{\substack{k=1\\k\neq i}}^{r}\max_{j\in M_k} -\langle a_j, x\rangle\right\} - \sum_{k=1}^{r}\max_{j\in M_k} -\langle a_j, x\rangle$$

and which completes the proof. □

In other words: For $f(x) = \max_{i\in I}\ \min_{j\in M_i} f_j(x)$ we have

$$\begin{aligned} v \quad \mapsto \quad df(x_0, v) = \left.\frac{df}{dv}\right|_{x_0} &= \lim_{\substack{\alpha\to 0\\ \alpha>0}}\frac{f(x_0+\alpha v) - f(x_0)}{\alpha} \in \mathrm{DCH}(\mathbb{R}^n) \\ &= \{h = p - q \mid p, q : \mathbb{R}^n \to \mathbb{R},\quad \text{sublinear}\} \end{aligned}$$

which can be considered as a starting point for studying *quasidifferentiable functions*.

3.2. The quasidifferential calculus

Let $(X, \|\cdot\|)$ be a normed vector space $X^{'}$ its dual space endowed with the weak-*-topology $\sigma(X, X^{'})$ and $U \subseteq X$ be an open subset of X. The dual norm of X will be denoted by $\|\cdot\|^{'}$ and $\langle\cdot,\cdot\rangle\ :\ X^{'}\times X \to \mathbb{R}$ stands for the dual pairing. It follows from the Theorem of Alaoglu–Bourbaki that the elements of $\mathcal{K}(X^{'})$ are bounded in the dual norm.

Definition 3 A function $f : U \to \mathbb{R}$ is said to be *quasidifferentiable* at $x_0 \in U$ if f is continuous at x_0 and if the following two conditions are satisfied:

a) For every $g \in X \setminus \{0\}$ the directional derivative

$$\left.\frac{df}{dg}\right|_{x_0} = \lim_{t\to 0+}\frac{f(x_0+tg) - f(x_0)}{t}$$

exists.

b) There exist two sets $\underline{\partial} f|_{x_0},\ \overline{\partial} f|_{x_0} \in \mathcal{K}(X^{'})$ called *sub- and superdifferential* such that

$$\begin{aligned} g \quad \mapsto \quad \left.\frac{df}{dg}\right|_{x_0} &= \max_{v\in\underline{\partial} f|_{x_0}}\langle v, g\rangle + \min_{w\in\overline{\partial} f|_{x_0}}\langle w, g\rangle \\ &= \max_{v\in\underline{\partial} f|_{x_0}}\langle v, g\rangle - \max_{w\in -\overline{\partial} f|_{x_0}}\langle w, g\rangle \in \mathrm{DCH}(X). \end{aligned}$$

The pair $\mathrm{D}f\big|_{x_0} = (\underline{\partial} f\big|_{x_0}, \overline{\partial} f\big|_{x_0})$ consisting of a sub- and superdifferential is called a *quasidifferential* of f at $x_0 \in U$.

Remark: Differing from the above notation of a quasidifferential

$$\mathrm{D}f\big|_{x_0} = (\underline{\partial} f\big|_{x_0}, \overline{\partial} f\big|_{x_0}),$$

which was introduced by V.F. Demyanov and A.M. Rubinov, we will use in this paper the equivalent notation

$$\left[\underline{\partial} f|_{x_0}, -\overline{\partial} f|_{x_0})\right] \in \mathcal{K}^2(X')/_{\sim}$$

because of its consistency with the Minkowski–Rådström–Hörmander approach.

It follows immediately from the definition of quasidifferentiability that every DC-function (difference of convex functions) is quasidifferentiable and by Proposition 2 every finite max–min combination of C^1-functions in finitely many variables is also quasidifferentiable. Relations between the quasidifferential and the Clarke subdifferential have been investigated in [3]. Moreover, there exists a complete calculus for quasidifferentiable functions (see [3]), which we summarize now:

Proposition 4 *Let $(X, \|\cdot\|)$ be a normed vector space and let $U \subseteq X$ be an open subset. Then the finite sums, products and quotients of quasidifferentiable functions are quasidifferentiable and every finite max–min combination of quasidifferentiable functions is again a quasidifferentiable function.*

Proof: Let $f, g : U \longrightarrow \mathbb{R}$ be two quasidifferentiable functions. By the rules for directional derivatives we have for fixed $x_0 \in U$ and direction $h \in X$:

$$\frac{d(f \pm g)}{dh}\bigg|_{x_0} = \lim_{\alpha \downarrow 0} \frac{[f(x_0 + \alpha h) \pm g(x_0 + \alpha h)] - [f(x_0) \pm g(x_0)]}{\alpha}$$

$$= \lim_{\alpha \downarrow 0} \frac{f(x_0 + \alpha h) - f(x_0)}{\alpha} \pm \lim_{\alpha \downarrow 0} \frac{g(x_0 + \alpha h) - g(x_0)}{\alpha} = \frac{df}{dh}\bigg|_{x_0} \pm \frac{dg}{dh}\bigg|_{x_0}.$$

Since $h \mapsto \frac{df}{dh}\Big|_{x_0}$, $h \mapsto \frac{dg}{dh}\Big|_{x_0} \in \mathrm{DCH}(X)$, it follows that $f \pm g$ is quasidifferentiable.

Since for every $t \in \mathbb{R}$

$$\frac{d(tf)}{dh}\bigg|_{x_0} = \lim_{\alpha \downarrow 0} \frac{tf(x_0 + \alpha h) - tf(x_0)}{\alpha} = t \lim_{\alpha \downarrow 0} \frac{f(x_0 + \alpha h) - f(x_0)}{\alpha} = t \frac{df}{dh}\bigg|_{x_0}$$

holds, it follows that the function tf quasidifferentiable at x_0. Analogously we have for

$$\frac{d(fg)}{dh}\bigg|_{x_0} = \lim_{\alpha \downarrow 0} \frac{[f(x_0 + \alpha h)g(x_0 + \alpha h)] - [f(x_0)g(x_0)]}{\alpha}$$

$$= \lim_{\alpha \downarrow 0} g(x_0 + \alpha h) \frac{f(x_0 + \alpha h) - f(x_0)}{\alpha} + \lim_{\alpha \downarrow 0} f(x_0) \frac{g(x_0 + \alpha h) - g(x_0)}{\alpha}$$

$$= g(x_0) \frac{df}{dh}\bigg|_{x_0} + f(x_0) \frac{dg}{dh}\bigg|_{x_0}$$

and hence by fg is quasidifferentiable at x_0.

If $g(x_0) \neq 0$, then

$$\frac{d(\frac{f}{g})}{dh}\Big|_{x_0} = \lim_{\alpha\downarrow 0}\frac{1}{\alpha}\left(\frac{f(x_0+\alpha h)}{g(x_0+\alpha h)} - \frac{f(x_0)}{g(x_0)}\right)$$

$$= \lim_{\alpha\downarrow 0}\frac{1}{\alpha}\left(\frac{f(x_0+\alpha h)g(x_0) - g(x_0+\alpha h)f(x_0)}{g(x_0)g(x_0+\alpha h)}\right)$$

$$= \lim_{\alpha\downarrow 0}\frac{1}{g(x_0+\alpha h)g(x_0)}\ \lim_{\alpha\downarrow 0}\frac{f(x_0+\alpha h)g(x_0) - g(x_0+\alpha h)f(x_0)}{\alpha}$$

$$= \frac{1}{g^2(x_0)}\ \lim_{\alpha\downarrow 0}\frac{[f(x_0+\alpha h)g(x_0) - f(x_0)g(x_0)] + [f(x_0)g(x_0) - g(x_0+\alpha h)f(x_0)]}{\alpha}$$

$$= \frac{1}{g^2(x_0)}\ \left[g(x_0)\lim_{\alpha\downarrow 0}\frac{[f(x_0+\alpha h) - f(x_0)]}{\alpha} - f(x_0)\lim_{\alpha\downarrow 0}\frac{[g(x_0+\alpha h) - g(x_0)]}{\alpha}\right]$$

$$= \frac{1}{g^2(x_0)}[g(x_0)\frac{df}{dh}\Big|_{x_0} - f(x_0)\frac{dg}{dh}\Big|_{x_0}].$$

Hence $\frac{f}{g}$ is quasidifferentiable at x_0.

Now we consider the function $\max\{f,g\}$. When $f(x_0) \neq g(x_0)$, the situation is clear, since for α small enough $\max\{f(x_0+\alpha h), g(x_0+\alpha h)\}$ is equal either $f(x_0+\alpha h)$ or $g(x_0+\alpha h)$ and one has only to consider one of the functions f or g. Now assume that $f(x_0) = g(x_0)$. Then we have

$$\frac{d(\max\{f,g\})}{dh}\Big|_{x_0} = \lim_{\alpha\downarrow 0}\frac{\max\{f(x_0+\alpha h), g(x_0+\alpha h)\} - f(x_0)}{\alpha}$$

$$= \max\left\{\lim_{\alpha\downarrow 0}\frac{f(x_0+\alpha h) - f(x_0)}{\alpha},\ \lim_{\alpha\downarrow 0}\frac{g(x_0+\alpha h) - g(x_0)}{\alpha}\right\}$$

$$= \max\ \left\{\frac{df}{dh}\Big|_{x_0}, \frac{dg}{dh}\Big|_{x_0}\right\}.$$

For the minimum we use the formula $\min\{f(x), g(x)\} = -\max\{-f(x), -g(x)\}$. □

3.3. The finite dimensional case

Now we restrict ourselves to the finite-dimensional case.

Theorem 5 *Let $(X, \|\cdot\|)$ be a finite-dimensional normed vector space, i.e. an Euclidean space with the standard topology, $U \subseteq X$ an open subset and $f : U \longrightarrow \mathbb{R}$ a locally Lipschitz function. Then f is quasidifferentiable at $x_0 \in U$ if and only if there exists an element $df\big|_{x_0} \in \mathrm{DCH}(X)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$, such that for all $h \in U$ with $\|h\| \leq \delta$ and $x_0 + h \in U$ the following inequality*

$$\left|f(x_0+h) - f(x_0) - df\big|_{x_0}(h)\right| \leq \varepsilon\|h\| \qquad (3..1)$$

holds.

Proof: The sufficiency is obvious. Now let f be a locally Lipschitz function which is quasidifferentiable. Then we have to show that the condition (3..1) is satisfied. Assume that this is not true. Then there exists a locally Lipschitz quasidifferentiable function, $f : U \to \mathbb{R}$, which does not satisfy condition (3..1). Hence there exists an $\varepsilon > 0$, such that for every $k \in \mathbb{N}$ there exists an element $h_k \in X \setminus \{0\}$ with $\|h_k\| \le \frac{1}{k}$ and

$$\left| f(x_0 + h_k) - f(x_0) - df\Big|_{x_0}(h_k) \right| \ge \varepsilon \|h_k\|.$$

Since $f : U \to \mathbb{R}$ is locally Lipschitz, there exists a ball $\mathbb{B}(x_0, r) \subseteq U$ and a real number $M > 0$, such that for all $x, y \in \mathbb{B}(x_0, r)$

$$\big|f(x) - f(y)\big| \le M\|x - y\|.$$

Now define: $g_k = r\frac{h_k}{\|h_k\|}$ and choose a convergent subsequence, also denoted by $(g_k)_{k\in\mathbb{N}}$, with $\lim g_k = g$ and $\lim \alpha_k = 0$, where $\alpha_k = \frac{\|h_k\|}{r}$. Then for all $k \in \mathbb{N}$ we have

$$\left| f(x_0 + \alpha_k g) - f(x_0) - df\Big|_{x_0}(\alpha_k g) \right|$$

$$= \Big| f(x_0 + \alpha_k g_k) + f(x_0 + \alpha_k g) - f(x_0 + \alpha_k g_k) - f(x_0)$$

$$-df\Big|_{x_0}(\alpha_k g) + df\Big|_{x_0}(\alpha_k g_k) - df\Big|_{x_0}(\alpha_k g_k) \Big|$$

$$= \Big| (f(x_0 + \alpha_k g_k) - f(x_0) - df\Big|_{x_0}(\alpha_k g_k))$$

$$-((f(x_0 + \alpha_k g_k) - f(x_0 + \alpha_k g)) - (df\Big|_{x_0}(\alpha_k g) - df\Big|_{x_0}(\alpha_k g_k)) \Big|$$

$$\ge \left| f(x_0 + \alpha_k g_k) - f(x_0) - df\Big|_{x_0}(\alpha_k g_k) \right|$$

$$-\left| (f(x_0 + \alpha_k g_k) - f(x_0 + \alpha_k g_k)) - (df\Big|_{x_0}(\alpha_k g_k) - df\Big|_{x_0}(\alpha_k g)) \right|$$

$$\ge \left| f(x_0 + \alpha_k g_k) - f(x_0) - df\Big|_{x_0}(\alpha_k g_k) \right|$$

$$-(\big|f(x_0 + \alpha_k g_k) - f(x_0 + \alpha_k g)\big| + \left| df\Big|_{x_0}(\alpha_k g_k) - df\Big|_{x_0}(\alpha_k g) \right|$$

$$\ge \varepsilon_0 \alpha_k r - (M\alpha_k\|g_k - g\| + L\alpha_k\|g_k - f\|).$$

$$= (\varepsilon_0 r - M\|g_k - g\| - L\|g_k - f\|)\alpha_k.$$

There exists an index $k_0 \in \mathbb{N}$ such that for all $k > k_0$

$$\varepsilon_0 r - M\|g_k - g\| - L\|g_k - f\| \ge \frac{\varepsilon_0 r}{2},$$

where L denotes the Lipschitz constant of $df\big|_{x_0}$. Hence for all $k \geq k_0$ we have

$$\left| \frac{f(x_0 + \alpha_k g) - f(x_0)}{\alpha_k} - df\big|_{x_0} \right| \geq \frac{\varepsilon_0 r}{2},$$

which implies that the directional derivative of the function f in direction g at the point $x_0 \in U$ does not exist. But this is a contradiction to the assumption, that $f : U \to \mathbb{R}$ is quasidifferentiable at $x_0 \in U$. $\square$

3.4. Local extrema, ascent and descent directions, examples

For a finite-dimensional spaces $X = \mathbb{R}^n$, the directions of steepest ascent and descent for a DCH-function, i.e. a pointwise difference of support functions, can be determined by solving a *"quadratic minimax"* problem (see [3]).

Let $X = \mathbb{R}^{\mathrm{n}}$ be equipped with the Euclidean norm $\|x\| = \sqrt{\langle x, x\rangle}$. The *steepest descent directions* of $\varphi \in \mathrm{DCH}(X)$ at the point $0 \in \mathbb{R}^{\mathrm{n}}$ are the vectors

$$\mathrm{Desc}(\varphi) = \left\{ x_0 \in X \mid \|x_0\| = 1 \ \text{ and } \ \varphi(x_0) = \inf_{\substack{x \in X \\ \|x\|=1}} \varphi(x) \right\}$$

and the *steepest ascent directions* of $\varphi = p_A - p_B \in \mathrm{DCH}(X)$ are the vectors

$$\mathrm{Asc}(\varphi) = \left\{ x_0 \in X \mid \|x_0\| = 1 \ \text{ and } \ \varphi(x_0) = \sup_{\substack{x \in X \\ \|x\|=1}} \varphi(x) \right\}.$$

Theorem 6 *Let $X = \mathbb{R}^{\mathrm{n}}$ be equipped with the Euclidean norm. Then for $\varphi = p_A - p_B = \max_{a \in A} \langle a, x\rangle - \max_{b \in B} \langle b, x\rangle \in \mathrm{DCH}(X)$ holds:*

i) $x_0 \in \mathrm{Desc}(\varphi)$ *if and only if*

$$x_0 = -\frac{w_0 + v_0}{\|w_0 + v_0\|} \quad \text{with} \quad \|w_0 + v_0\| = \sup_{w \in -B} \inf_{v \in A} \|w + v\|.$$

ii) $x_0 \in \mathrm{Asc}(\varphi)$ *if and only if*

$$x_0 = \frac{w_0 + v_0}{\|w_0 + v_0\|} \quad \text{with} \quad \|w_0 + v_0\| = \sup_{v \in -A} \inf_{w \in B} \|w + v\|,$$

where w_0 and v_0 are optimal solutions of the corresponding optimization problems.

Proof: It is sufficient to prove the formula only for the steepest descent directions since the proof for the steepest ascent directions follows exactly from the same calculation.

For $\varphi = p_A - p_B \in \mathrm{DCH}(X)$ and $x_0 \in \mathrm{Desc}(\varphi)$ there holds:

$$\begin{aligned}
\varphi(x_0) &= \inf_{\substack{x\in X\\ \|x\|=1}} \varphi(x) = \inf_{\substack{x\in X\\ \|x\|=1}} (p_A(x) - p_B(x)) \\
&= \inf_{\substack{x\in X\\ \|x\|=1}} \left(\sup_{v\in A}\langle v, x\rangle - \sup_{w\in B}\langle w, x\rangle \right) = \inf_{\substack{x\in X\\ \|x\|=1}} \left(\inf_{w\in -B} \sup_{v\in A}\langle v+w, x\rangle \right) \\
&= \inf_{w\in -B} \left(\inf_{\substack{x\in X\\ \|x\|=1}} \sup_{v\in A}\langle v+w, x\rangle \right) = \inf_{w\in -B} \left(\sup_{v\in A} \inf_{\substack{x\in X\\ \|x\|=1}} \langle v+w, x\rangle \right) \\
&= \inf_{w\in -B} \left(\sup_{v\in A} -\langle v+w, \frac{v+w}{\|v+w\|}\rangle \right) = \inf_{w\in -B} \left(-\inf_{v\in A} \|v+w\| \right) \\
&= -\sup_{w\in -B} \left(\inf_{v\in A} \|v+w\| \right).
\end{aligned}$$

The proof of the theorem follows immediately from this calculation. □

An obvious consequence of Theorem 6 are the following necessary optimality conditions for quasidifferentiable functions:

Proposition 7 *Let $U \subset \mathbb{R}^n$ be an open set, $x_0 \in U$ and $f : U \longrightarrow \mathbb{R}$ a quasidifferentiable function with the quasidifferential* $\mathrm{D}f\big|_{x_0} = (\underline{\partial} f\big|_{x_0}, \overline{\partial} f\big|_{x_0})$.

i) *If f has in $x_0 \in U$ a local maximum, then $-\overline{\partial} f\big|_{x_0} \subseteq \underline{\partial} f\big|_{x_0}$.*

ii) *If f has in $x_0 \in U$ a local minimum, then $-\underline{\partial} f\big|_{x_0} \subseteq \overline{\partial} f\big|_{x_0}$.*

Remark: In the notation of the Minkowski–Rådström–Hörmander space holds that if f has a local extremum at $x_0 \in U$, then the class $\left[\underline{\partial} f\big|_{x_0}, -\overline{\partial} f\big|_{x_0}\right] \in \mathcal{K}^2((\mathbb{R}^n)')\big/_{\sim}$ is convex.

Example 8 Typical quasidifferentiable functions are algebraic expressions of max–min combinations of convex- and C^1-functions. The following untypical example of a quasidifferentiable function is from S. Rolewicz [18]. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

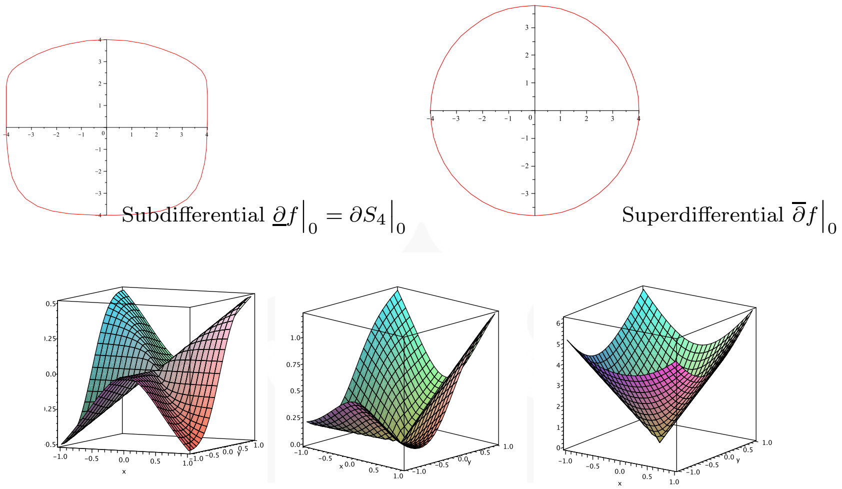
$$f(x_1, x_2) = \begin{cases} \frac{x_1^2 x_2}{x_1^2 + x_2^2} & \text{if} \quad (x_1, x_2) \neq (0,0), \\ 0 & \text{if} \quad (x_1, x_2) = (0,0). \end{cases}$$

Obviously f is everywhere differentiable except of $(0,0)$. Since f is positively homogeneous, it follows that $\left.\frac{df}{dh}\right|_0 = \lim\limits_{\substack{\alpha\to 0\\ \alpha>0}} \frac{f(0+\alpha g) - f(0)}{\alpha} = f(h)$. Now the following representation of $h \mapsto \left.\frac{df}{dh}\right|_0 f = f(h)$ as a pointwise difference of support functions is possible (see [13], Corollary 8.1.7): Let $\|\cdot\|_2$ be the Euclidean norm and define

$$S_\alpha(h) = \alpha\|h\|_2 + f(h)$$

for some $\alpha > 0$. First observe that the restriction of S_α is convex along every line through the origin. Furthermore observe that S_α is a smooth function outside the origin. Next, a lengthly but straightforward calculation shows, that for $\alpha = 4$ the Hessian of S_4 is positive semi-definite at any point outside the origin and hence

$S(h) = 4 \cdot \|h\|_2 + f(h)$ is positively homogeneous and convex, hence a support function.



Subdifferential $\underline{\partial} f\big|_0 = \partial S_4\big|_0$ Superdifferential $\overline{\partial} f\big|_0$

Function f Function $S_{0.5}$ Function S_4

Figure 1.:

Therefore, $\frac{df}{dh}\Big|_0 = f(h) = S(h) - 4 \cdot \|h\|_2$ which means that f is quasidifferentiable at every point of $\mathbb{R}^2$. In Fig. 1 the quasidifferential of f at the origin is shown together with the functions $f, S_{0.5}$ and S_4.

Form the two optimization problems

$$\max / \min \; x_1^2 x_2 \quad \text{under} \quad x_1^2 + x_2^2 = 1$$

follows, that the two steepest descent directions of f are: $(-\frac{1}{3}\sqrt{6}, -\frac{1}{3}\sqrt{3})$ and $(\frac{1}{3}\sqrt{6}, -\frac{1}{3}\sqrt{3})$ and that the two steepest ascent directions of f are: $(-\frac{1}{3}\sqrt{6}, \frac{1}{3}\sqrt{3})$ and $(\frac{1}{3}\sqrt{6}, \frac{1}{3}\sqrt{3})$ (cf. also Theorem 6 and Fig. 1).

4. Minimal pairs of compact convex sets

Now we state two typical sufficient conditions for minimality of pairs of compact convex sets. The first type of criteria uses conditions which ensure that two compact convex sets are in a certain 'general position', while the second type of criteria uses information about exposed points of the Minkowski sum of compact convex sets.

Theorem 9 *Let X be a topological vector space, A a polytope and $B \in \mathcal{K}(X)$. Furthermore, let us assume that A has k faces $S_1 = H_{f_1}(A), ..., S_k = H_{f_k}(A)$ of maximal dimension and that for every $i \in \{1, ..., k\}$ we have $H_{f_i}(B) = \{b_i\}$. Then the pair $(A, B) \in \mathcal{K}^2(X)$ is minimal.*

Proof: Let us assume that there exists a pair $(A', B') \in \mathcal{K}^2(X)$ with $(A', B') \sim (A, B)$ and $A' \subset A$ and $B' \subset B$. Then it follows from $A + B' = B + A'$ that for all $i \in \{1, ..., k\}$,
$S_i + H_{f_i}(B') = b_i + H_{f_i}(A')$ holds. Now let us choose elements $b_i^{'} \in H_{f_i}(B')$, $i \in \{1, ..., k\}$. Then we have for every $i \in \{1, ..., k\}$ that $S_i + b_i^{'} \subset b_i + H_{f_i}(A')$. Now put $x_i = b_i^{'} - b_i$. Then for every $i \in \{1, ..., k\}$ the inclusion $S_i + x_i \subset H_{f_i}(A') \subset A' \subset A$ holds.

Since the polytope A has k-faces $S_1 = H_{f_1}(A), ..., S_k = H_{f_k}(A)$ of maximal dimension it can be described in terms of inequalities as

$$A = \{x \in X \mid f_i(x) \leq \alpha_i \ , \ i \in \{1, ..., k\}\}$$

for some $\alpha_1, ..., \alpha_k \in \mathbb{R}$. Hence there exist real numbers $\beta_1, ..., \beta_k \in \mathbb{R}$ such that $\bar{A} = \{x \in X \mid f_i(x) \leq \beta_i \ , \ i \in \{1, ..., k\}\} \subset A$ and that for every $i \in \{1, ..., k\}$ we have $S_i + x_i = H_{f_i}(A) + x_i \subset H_{f_i}(\bar{A})$. Now from $\bar{A} \subset A$ it follows that for every $i \in \{1, ..., k\}$ the inequality $\beta_i \leq \alpha_i$ holds. Since every functional f_i, $i \in \{1, ..., k\}$ determines a face of maximal dimension of A it follows from the condition $S_i + x_i = H_{f_i}(A) + x_i \subset H_{f_i}(\bar{A})$ that for every $i \in \{1, ..., k\}$ the inequality $\alpha_i \leq \beta_i$ holds, which means that $\bar{A} = A$. Hence the pair $(A, B) \in \mathcal{K}^2(X)$ is minimal. □

Let $X = \mathbb{R}^n$ and P be a polytope. Then the *polar polytope* is defined by

$$P^o = \left\{ u \in \mathbb{R}^n \mid \sup_{x \in P} \langle u, x \rangle \leq 1 \right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^n$.

From the well known correspondence between extreme points of P and faces of maximal dimension of P^o we get:

Corollary 10 *Let $X = \mathbb{R}^n$ be the Euclidian space with standard topology and P be a polytope. Then the pair*

$$(P, P^o) \in \mathcal{K}^2(\mathbb{R}^n)$$

is minimal.

The next criterium for minimality is based on a sufficient condition on the indecomposability of a nonempty compact convex set.

Theorem 11 *Let $(X, \|\cdot\|)$ be a real Banach space, and let $(A, B) \in \mathcal{K}^2(X)$. If for every exposed point $a + b \in \mathcal{E}_0(A + B)$ with $a \in \mathcal{E}_0(A)$, $b \in \mathcal{E}_0(B)$ there exists $b_1 \in \mathcal{E}_0(B)$ or $a_1 \in \mathcal{E}_0(A)$ such that $a + b_1 \in \mathcal{E}_0(A + B)$ and $a - b_1 \in \mathcal{E}(A - B)$ or $a_1 + b \in \mathcal{E}_0(A + B)$ and $a_1 - b \in \mathcal{E}(A - B)$, then (A, B) is minimal.*

Proof: Let $(A,B) \in \mathcal{K}^2(X)$ and $f \in X^{'}$. Then $H_f(A+B) = H_f(A) + H_f(B)$. This implies the unique representation of every exposed point of $A+B$ as a sum of exposed points of A and B.

Let us show that the pair $(A,B) \in \mathcal{K}^2(X)$ is minimal. To do this, we choose a pair $(A^{'},B^{'}) \in \mathcal{K}^2(X)$ such that $A^{'} \subseteq A$, $B^{'} \subseteq B$ and $A + B^{'} = B + A^{'}$.

Let $a{+}b \in \mathcal{E}_0(A{+}B)$. Without loss of generality we can assume that for $a \in \mathcal{E}_0(A)$ there exists $b_0 \in \mathcal{E}(B)$ such that $a + b_0 \in \mathcal{E}_0(A+B)$ and $a - b_0 \in \mathcal{E}(A-B)$. Hence there exists a continuous linear functional $f_0 \in X^*$ such that $H_{f_0}(A{+}B) = \{a{+}b_0\}$. By the above formula for faces we have

$$H_{f_0}(A) = \{a\} \quad \textit{and} \quad H_{f_0}(B) = \{b_0\}.$$

Since $A + B^{'} = B + A^{'} =: K$, it follows that $H_{f_0}(A) + H_{f_0}(B^{'}) = H_{f_0}(B) + H_{f_0}(A^{'})$. Hence there exist elements $a^{'} \in H_{f_0}(A^{'}) \subseteq A$ and $b^{'} \in H_{f_0}(B^{'}) \subseteq B$ such that $a + b^{'} = b_0 + a^{'}$.

Since $a - b_0 \in \mathcal{E}(A-B)$, it follows that $a = a^{'}, b_0 = b^{'}$. The equality implies $a = a^{'}$ and $B + a \subseteq B + A^{'} = K$. Hence $a + b \in K$. Since $a + b \in \mathcal{E}_0(A+B)$, by a modification of V. Klee ([14], Theorem 2.4.4) of Krein–Milman theorem, follows that $A + B = K$.

Hence by the order cancellation law $A + B^{'} = B + A^{'}$ implies $A = A^{'}$ and $A + B^{'} = B + A^{'}$, $B = B^{'}$. Therefore $(A,B) \in \mathcal{K}^2(X)$ is minimal. □

We illustrate this criteria by the following two examples:

Example 12 Let us consider the 'Star of David' which is a pair of polar equilateral triangles in the plain and can be defined as follows:

For a positive real number R put $x = \frac{1}{2}\sqrt{3}R$, $y = \frac{1}{2}R$, $a_1 = (0,R)$, $a_2 = (x,-y)$, $a_3 = (-x,-y)$ and let $A = a_1 \vee a_2 \vee a_3$ and $B = -A$.

By Corollary 10 the pair $(A,B) \in \mathcal{K}^2(X)$ is minimal and the corresponding support functions are $p_A, p_B : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with :

$$p_A(x_1,x_2) = \max\left\{Rx_2, \frac{1}{2}\sqrt{2}Rx_1 - \frac{R}{2}x_2, -\frac{1}{2}\sqrt{2}Rx_1 - \frac{R}{2}x_2\right\}$$

and

$$P_B(x_1,x_2) = \max\left\{-Rx_2, -\frac{1}{2}\sqrt{2}Rx_1 + \frac{R}{2}x_2, \frac{1}{2}\sqrt{2}Rx_1 + \frac{R}{2}x_2\right\}.$$

In Figure 2 the pair (A,B) and the corresponding pointwise difference of support functions is depicted.

Example 13 Let us consider the pair $(A,B) \in \mathcal{K}^2(\mathbb{R}^2)$ of orthogonal lenses which is defined as follows:

Let $R > 0$ be given and consider the Euclidean balls

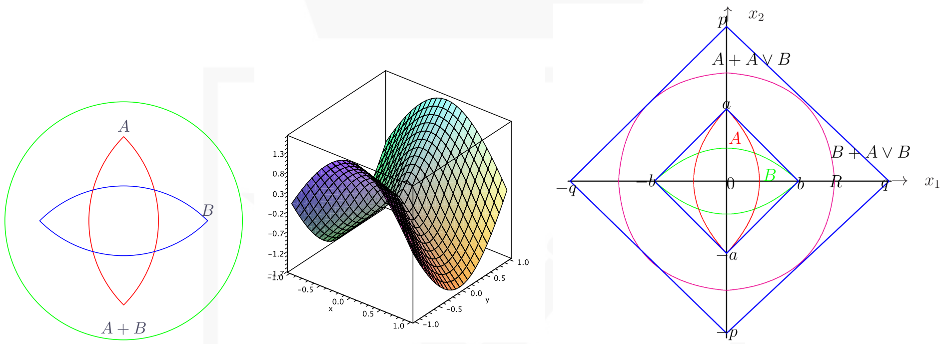
$$K_1 = \mathbb{B}\left(\left(\frac{1}{2}\sqrt{2}R, 0\right), R\right), \quad K_2 = \mathbb{B}\left(\left(-\frac{1}{2}\sqrt{2}R, 0\right), R\right)$$

in the plane $\mathbb{R}^2$. Furthermore, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x_1,x_2) = (-x_2, x_1)$. Put $A = K_1 \cap K_2$ and $B = T(A)$.

Then $A + B = A - B = B(0;R)$ is the ball with radius R at the origin $0 \in \mathbb{R}^2$ and it follows from Theorem 11 that the pair (A,B) is minimal.

Star of David $(A, B) \in \mathcal{K}^2(X)$ $\qquad$ $p_A - p_B$

Figure 2.:



Orthogonal lenses $\qquad$ $p_A - p_B$ $\qquad$ Function $A + A \vee B = \mathbb{B}(0, R) \vee p \vee \{-p\}$

Figure 3.:

To compute the corresponding pointwise difference of two support functions p_A and p_B put $a = \left(0, \frac{1}{2}\sqrt{2}R\right)$, $b = \left(\frac{1}{2}\sqrt{2}R, 0\right)$ and $p = (0, \sqrt{2}R)$, $q = (\sqrt{2}R, 0)$. Now observe that $(A, B) \sim (A + A \vee B, B + A \vee B)$ and that $A + A \vee B = \mathbb{B}(0, R) \vee p \vee \{-p\}$ (see Fig. 3).

Hence $p_A(x_1, x_2) = \max\left\{\sqrt{2}R|x_2|, R\sqrt{x_1^2 + x_2^2}\right\}$ and $p_B(x_1, x_2) = \max\left\{\sqrt{2}R|x_1|,\right.$ $\left. R\sqrt{x_1^2 + x_2^2}\right\}$.

4.1. Further properties of pairs of bounded closed convex sets

In addition to the above mentioned properties of pairs of compact convex sets in quasidifferential calculus, a further look to pairs of bounded closed convex sets reveals an interesting mathematical theory. As an example we consider the possibility of separating convex sets by convex sets. Formally this goes as follows: Let $A, B, S \in \mathcal{B}(X)$, then we say that S *separates* the sets A and B if for every $a \in A$ and $b \in B$ we have $[a, b] \cap S \neq \emptyset$, where $[a, b] = \{a\} \vee \{b\}$ the line segment between a and b.

The definition is illustrated in Fig. 4 for the convex sets $A, B, S \subseteq \mathbb{R}^2$. Now the

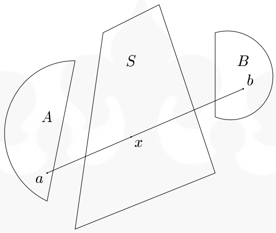


Figure 4.:

following statements are equivalent:

i) $A \cup B$ is convex, *ii)* $A \cap B$ separates A and B, *iii)* $A \vee B$ is a summand of $A \dot{+} B$.

This leads to the following algebraic characterization of 'separation': The condition that for $A, B, S \in \mathcal{B}(X)$ the inclusion $A + B \subset (A \vee B) \dot{+} S$ holds is equivalent to the statement that S separates the sets A and B. This formula is called the *separation law* and is an algebraic characterization of 'separation'. Moreover it turns out, that the separation law is equivalent to the order cancellation law.

This interplay between geometry and algebra is typical for the theory of pairs of bounded closed convex sets.

Further properties concern the 'translation invariance' for certain classes of minimal pairs of bounded closed convex sets, to which in particular belong the 'reduced' pairs which where introduced by Ch. Bauer and R. Schneider (see [2]). A pair $(A, B) \in \mathcal{B}^2(X)$ is called *reduced* if for any $(C, D) \in [A, B]$ there exits an $M \in \mathcal{B}(X)$ such that $C = A \dot{+} M$ and $D = B \dot{+} M$ holds.

Beside further properties about 'conditional minimality', 'invariant convexificators' and the 'invariance of dimension' for minimal pairs, let us finally mention the 'semigroup property' of $\mathcal{B}(X)$. Namely it is possible to consider the theory of pairs of bounded closed convex sets in the more general frame of a commutative semigroup S which is ordered by a relation $\leq$ and which satisfies the order cancellation law, i.e.: if $as \leq bs$ for some $s \in S$, then $a \leq b$. Within this frame $(a, b) \in S^2 = S \times S$ corresponds to a *fraction* $a/b \in S^2$ and *minimality* to a relative prime representation of $a/b \in S^2$.

For more details we refer to the work of J. Grzybowski et al. [5, 6, 7, 14].

Independently of this work pairs of polytopes have also been extensively studied in literature as ’virtual polytopes’ in the framework of combinatorial convexity and algebraic geometry. In this connection we refer to the book of G. Ewald [4] as well as the paper of R. Langevin, G. Levitt and H. Rosenberg [12] and A.V. Pukhlikov and A.G. Khovanskiĭ [17]. More recent publications on this topic are the papers of G. Panina [15] and M. Knyazeva and G. Panina [10].

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