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An Experimental Comparison of Popular Estimation Methods for the Weibull, Gamma and Gompertz Distributions

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Abstract. The aim of this study was to describe some parametric estimation methods for the Weibull, gamma and Gompertz distributions and to identify among them estimators the most efficient in practical applications. Techniques which are considered as traditional methods, like the maximum likelihood (MLE) and the method of moments (MM) estimation but also some newer and less commonly used techniques like the L-moment estimator (LME), least-square estimator (LSE), generalized spacing estimator (GSE) and percentile estimator (PE) were presented. The application of each method was demonstrated in a simulation study using data sets generated for different distribution parameters and sample sizes. Discussed estimators were compared in terms of their efficiency and bias measured by mean-square errors (MSE) based on the simulations results.

Keywords: parameter estimation, Weibull distribution, gamma distribution, Gompertz distribution, MSE.

1. Introduction

The two-parameter Weibull, gamma and Gompertz probability distributions have many useful applications in areas of the technology and natural sciences (especially in failure and survival analysis). Therefore the proper estimation of their parameters is a very important and wide discussed problem.

The maximum likelihood (MLE) and the method of moments (MM) estimation are nowadays traditional statistical methods. The MLE is the most common-used estimator for its efficiency and good theoretical properties, while the method of moments is easily applicable and often gives explicit algebraic estimates for a considered probability distribution. However, in some special cases, these traditional estimators are ineffective in terms of statistical or computation properties. In particular, for some (e.g. the Weibull and the Gompertz) distributions even moment estimators do not provide explicit estimates of unknown parameters and require application of the numerical methods.

Therefore, other methods have been proposed in statistical literature as an alternative to these traditional techniques. Among them methods such as the L-moment estimator (LME), least-square estimator (LSE), generalized spacing estimator (GSE) and percentile estimator (PE) are often suggested. Generally, these methods do not present better theoretical properties than the well-known MLE and MM methods, but in special cases they could be a better approximation of unknown parameters.

The aim of this study was to describe some parametric estimation methods for the Weibull, gamma and Gompertz distributions and to identify in a simulation study the most efficient estimator among them. The application of each method was demonstrated for discussed probability distributions using data sets generated by the R statistical software for different values of distribution parameters and data sample sizes. The estimators were compared in terms of their efficiency and bias measured by the mean-square error (MSE).

2. The Weibull, gamma and Gompertz distribution

The Weibull, gamma and Gompertz distributions are parametric distributions commonly used in practical applications for their flexibility and good fit to survival and failure data. These two-parameter distributions have increasing, decreasing or stable failure rates depending on the shape parameter value. On the other hand the scale parameter determines a spread of the distribution. Density functions $f = f(x, \lambda, k)$ for the Weibull, gamma and Gompertz distributions are given as in Tab. 1.

Tab. 1. Density functions of the Weibull, gamma and Gompertz distributions

Distribution	$f(x, \lambda, k)$
Weibull	$\frac{k}{\lambda} \cdot \left(\frac{x}{\lambda}\right)^{k-1} \cdot e^{-\left(\frac{x}{\lambda}\right)^k}, \; x \geqslant 0, \; \lambda > 0, \; k > 0$
gamma	$x^{k-1} \cdot \frac{e^{-\frac{x}{\lambda}}}{\Gamma(k)\cdot\lambda^k}, \; x \geqslant 0, \; \lambda > 0, \; k > 0$
Gompertz	$\lambda e^{kx} \cdot \exp\left(-\frac{\lambda}{k} \cdot \left(e^{kx} - 1\right)\right), \; x \geqslant 0, \; \lambda > 0, \; k > 0$

In the above expressions and the rest of this paper λ and k denote the scale of the shape distribution parameters, respectively.

3. Description of the estimators

In this section I give a brief description of the parameter estimators used in the study. In the next section I present simulation results and comparison of estimators for different values of the distribution parameters and data sample sizes. Concluding remarks are given in Section 5.

Most of the estimation methods considered in this study do not provide explicit estimates of the distribution parameters and their application requires numerical calculations. In such cases the Newton-Raphson's (for univariate equations) or the Nelder-Mead's (for multivariate equations) methods were applied in the study.

3.1. The maximum likelihood estimation method

The maximum likelihood estimation method had been used in special cases by Gauss in 1812 but a full description of properties and a presentation of its application were performed 100 years later by Ronald Fisher [1]. Nowadays, the maximum likelihood method is the most popular estimation technique, mainly for its good theoretical properties. See other literature [5, 12, 18, 20] for the existence and the uniqueness of the maximum likelihood estimates for discussed distributions.

The idea of the maximum likelihood method is based on the assumption that observed data are the most likely outcome of a random experiment in respect to the considered probability distribution. In the discussed method the key role plays the likelihood function specified as the probability of observed data depending on the values of distribution parameters. In case of the Weibull, gamma and the Gompertz distributions the likelihood function L is given as in Tab. 2.

Tab. 2. The likelihood functions L for the Weibull, gamma and Gompertz distributions

Distribution	$L\,(k, \lambda\,, x_1, \ldots, x_n)$
Weibull	$\left(\frac{k}{\lambda}\right)^n \prod_{i=1}^n \left(\frac{x_i}{\lambda}\right)^{k-1} \cdot e^{-\sum\limits_{i=1}^{n}\left(\frac{x_i}{\lambda}\right)^k}$
gamma	$\frac{1}{\Gamma(k)^n \cdot \lambda^n} \prod_{i=1}^n \left(\frac{x_i}{\lambda}\right)^{k-1} \cdot e^{-\sum\limits_{i=1}^{n} \frac{x_i}{\lambda}}$
Gompertz	$\lambda^n \cdot \exp\left(k \sum\limits_{i=1}^{n} x_i - \frac{\lambda}{k} \cdot \left(\sum\limits_{i=1}^{n} e^{kx_i} - n\right)\right)$

The maximum likelihood estimators of the distribution parameters are found by maximizing the likelihood functions L (actually their logarithms) with respect to parameter values. Maximum likelihood estimates of the shape and scale parameters for the Weibull, gamma and Gompertz distributions are therefore the solutions of equations (which are obtained by equating partial derivatives of $\ln(L)$ to zero) presented in Tab. 3.

Tab. 3. Equations specifying the MLE estimates for the Weibull, gamma and Gompertz distributions

Distribution	MLE estimates
Weibull	$n \cdot \sum_{i=1}^{n} x_i^{\hat{k}} \cdot \ln(x_i) = \sum_{i=1}^{n} x_i^{\hat{k}} \cdot \left(\frac{n}{\hat{k}} + \sum_{i=1}^{n} \ln(x_i) \right), \quad \hat{\lambda} = \sqrt[\hat{k}]{\frac{1}{n} \sum_{i=1}^{n} x_i^{\hat{k}}}$
gamma	$\bar{x} = \hat{k} \cdot \sqrt[n]{\Pi_{i=1}^{n} x_i} \cdot \exp\left(-\frac{\cdot\Gamma'(\hat{k})}{\Gamma(\hat{k})} \right), \quad \hat{\lambda} = \frac{\bar{x}}{\hat{k}}$
Gompertz	$\left(\sum_{i=1}^{n} e^{\hat{k}x_i} - n \right) \cdot \left(\hat{k}\bar{x} + 1 \right) = \hat{k} \cdot \sum_{i=1}^{n} x_i e^{\hat{k}x_i}, \quad \hat{\lambda} = \frac{n\hat{k}}{\sum_{i=1}^{n} e^{\hat{k}x_i - n}}$

Discussed problems have no explicit algebraic solutions, therefore numerical calculations are required. In the simulation study for the maximum likelihood estimation the Newton-Raphson's method was applied.

3.2. The method of moments estimation

The method of moments estimation is based on a simple observation that for a large enough data sample drawn from a given probability distribution empirical and theoretical moments have asymptotically equal values. In mathematical consideration this remark is known as the strong law of large numbers.

In Tab. 4 the selected theoretical moments of the Weibull, gamma and Gompertz distributions are given,

Tab. 4. Theoretical moments of the Weibull, gamma and Gompertz distributions

Distribution	Theoretical moments
Weibull	$\mu_1 = \lambda \cdot \Gamma\left(1 + \frac{1}{k}\right), \quad \mu_2 = \lambda^2 \cdot \Gamma\left(1 + \frac{2}{k}\right)$
gamma	$\mu_1 = k\lambda, \quad \sigma_2^2 = k\lambda^2$
Gompertz	$\mu_1 = \frac{1}{k} \cdot e^{\frac{\lambda}{k}} \cdot \Gamma\left(0, \frac{\lambda}{k}\right), \quad \mu_2 = e^{\frac{\lambda}{k}} \cdot \frac{1}{k^2} \cdot \int_{\frac{\lambda}{k}}^{\infty} \left(\ln u - \ln\left(\frac{\lambda}{k}\right)\right)^2 \cdot e^{-u} du$

where μ_k denotes raw k-moments and σ_2^2 the variance of the probability distribution. Starting from equalities of specified theoretical moments and their empirical analogues (respectively m_k and s_2^2) we obtain estimates of parameters for the Weibull, gamma and Gompertz distributions. The results of the calculations are presented in Tab. 5.

The method of moments estimation provides an explicit algebraic solution only for the gamma distribution. In the rest of the analyzed distributions, similarly as in the situation discussed for the MLE, estimation of unknown parameters requires application of numerical calculations.

Tab. 5. The MM etimates for the Weibull, gamma and Gompertz distributions

Distribution	MM estimates
Weibull	$\frac{\Gamma\left(1+\frac{2}{k}\right)}{\Gamma^2\left(1+\frac{1}{k}\right)} = \frac{\left(s_2^2+m_1^2\right)}{m_1^2}, \ \hat{\lambda} = \frac{m_1}{\Gamma\left(1+\frac{1}{k}\right)}$
gamma	$\hat{\lambda} = \frac{s_2^2}{m_1}, \quad \hat{k} = \frac{m_1^2}{s_2^2}$
Gompertz	if x is a solution of $\frac{m_2}{{m_1}^2} \cdot e^x \cdot \Gamma^2(0,x) = \int_x^\infty e^{-u} \left(\ln u - \ln x\right)^2 du$, then $\hat{k} = e^x \cdot \frac{\Gamma(0,x)}{m_1}, \ \hat{\lambda} = \hat{k} \cdot x$

3.3. The L-moments method estimation

The L-moments method was suggested in 1990 by British mathematician J. Hosking [10] and it is one of the series of estimation techniques based on order statistics.

Let X be a random variable and $X_{1:n}, \ldots, X_{n:n}$ be the order statistics of a random sample drawn from a continuous distribution of X. We consider one of the characteristics of variable X called the L-moments and define it as:

$$\lambda_r = \frac{1}{r} \cdot \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} EX_{r-k:r}, \quad r = 1, \ldots, n.$$

Particularly, for continuous probability distributions there are:

$$\lambda_1 = EX, \quad \lambda_2 = \int_{\mathbb{R}} x \cdot (2f(x)F(x) - f(x))\, \mathrm{d}x.$$

L-moments are the coefficients of the shifted Legendre series determined for the quantile function of X. The shifted Legendre polynomials are orthogonal on interval $(0,1)$ with a constant weight function. Hosking proved [10] that L-moments of a real-valued random variable X exist if and only if X has a finite mean. In that case the distribution of X is characterized by a set of its L-moments. The natural definition of the empirical L-moments for the observed data $x_1, \ldots, x_n$ is as follows:

$$l_r = \frac{1}{\binom{n}{r}} \cdot \sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_r \leqslant n} \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} x_{i_{r-k}:n}, \quad r = 1, \ldots, n.$$

Comparing theoretical and sample L-moments for the Weibull, gamma and Gompertz distributions we get estimates of k and λ parameters given as solutions of equations presented in Tab. 6.

See [10] for more details about theoretical properties and examples of the L-moments applications in practice.

Tab. 6. The equations specifying the LME estimates for the Weibull, gamma and Gompertz distributions

Distribution	LME estimates
Weibull	$\hat{k} = \frac{-\ln(2)}{\ln\left(1-\frac{l_2}{l_1}\right)}, \quad \hat{\lambda} = \frac{l_1}{\Gamma\left(1+\frac{1}{\hat{k}}\right)}$
gamma	$\frac{l_2}{l_1} = \frac{1}{\sqrt{\pi}} \frac{\Gamma\left(\hat{k}+\frac{1}{2}\right)}{\hat{k}\Gamma(\hat{k})}, \quad \hat{\lambda} = \frac{l_1}{\hat{k}}$
Gompertz	if x is a solution of $\frac{l_1}{l_1-l_2} = \frac{\gamma(0,x)}{e^x \cdot \gamma(0,2x)}$, then $\hat{k} = \frac{e^x \cdot \gamma(0,x)}{l_1}, \quad \hat{\lambda} = x \cdot \hat{k}$

3.4. The least-square estimation method

Suppose that $X_1, \ldots, X_n$ is a random sample with continuous distribution function F_θ and $X_{1:n}, \ldots, X_{n:n}$ is the order statistics. Theoretical mean values of random variables obtained by transformation of order statistics by distribution function F_θ are following:

$$E\left(F_\theta\left(X_{i:n}\right)\right) = \frac{i}{n+1}, \qquad i = 1, \ldots, n.$$

The least-square estimation (LSE) method is based on a fact that the empirical and theoretical mean values of the variables $F_\theta(X_{i:n})$ have for a large enough sample size asymptotically equal values (for each $i = 1, \ldots, n$). In the LSE method the distance between that values is determined by the Euclidean measure in $\mathbb{R}^n$. The initially considered estimation problem could be therefore transformed to the minimization problem of the following expression:

$$\sum_{i=1}^{n} \left(F_\theta(x_{i:n}) - \frac{i}{n+1} \right)^2 .$$

Minimization of this expression for discussed distributions is very complicated and requires many calculations. However, in some cases transformation of the minimization problem to a linear form allows to easily get a solution. The least-square estimators determined for the Weibull distribution after linearization of the LSE problem are given as:

$$\hat{k} = \frac{\sum_{i=1}^{n} y_i x_i - n\bar{x}\bar{y}}{\sum_{i=1}^{n} (x_i - \overline{x})^2}, \quad \hat{\lambda} = \overline{y} - \hat{k}\overline{x}$$

and for the Gompertz distribution, if $\hat{b}$ is a solution of:

$$\frac{\sum_{i=1}^{n} y_i \cdot (1 - e^{\hat{b}x_i})}{\sum_{i=1}^{n} (1 - e^{\hat{b}x_i})^2} = \frac{\sum_{i=1}^{n} y_i x_i \cdot e^{\hat{b}x_i}}{\sum_{i=1}^{n} x_i \cdot e^{\hat{b}x_i} (1 - e^{\hat{b}x_i})},$$

then

$$\hat{\lambda} = \hat{b}, \quad \hat{k} = \frac{\sum_{i=1}^{n} y_i \cdot (1 - e^{\hat{b}x_i})}{\sum_{i=1}^{n} (1 - e^{\hat{b}x_i})^2} \cdot \hat{b}.$$

Nature of the gamma distribution prevents application of linearization. Usage of the LSE method in that case requires many numerical computations and is exposed to a significant bias. Therefore in the estimation of the gamma distribution parameters the LSE method was not considered.

In the above calculations we considered approximation of $F_\theta(x_{i:n})$ by mean values. In some statistical studies also another possible approximations were proposed, e.g. median values or adjusted mean values ranks as follows:

- $p_i = \frac{i-0{,}5}{n}$ (*an adjusted mean rank*),
- $p_i = \frac{i-0{,}3}{n+0{,}4}$ (*a median rank*),
- $p_i = \frac{i-\frac{3}{8}}{n+\frac{1}{4}}$ (*a symmetrical rank*).

To compare different forms of $F_\theta(X_{i:n})$ values approximation, in terms of their influence on final estimation results, in the simulation study the mean values and the median values ranks (the most popular ranks in statistical literature) were considered. Simulation results for these values are denoted in the paper by LSE1 and LSE2, respectively. More details about the problem of the proper choice of the approximation values for $F_\theta(X_{i:n})$ variables are available in literature [3, 6].

3.5. The generalize spacing method estimation

Let $X_1, \ldots, X_n$ be a random sample drawn from a continuous distribution with distribution function F_θ, and $X_{1:n}, \ldots, X_{n:n}$ be an order sample. The generalize spacing estimation (GSE) method originally proposed by R. Cheng and N. Amin [4] and independently by B. Ranneby [19] is based on a concept of ”spacings” defined as follows:

$$D_i\left(\theta\right) = F_\theta\left(X_{i:n}\right) - F_\theta\left(X_{i-1:n}\right), \;\; i = 1, \ldots, n+1,$$

where $F_\theta(X_{0:n}) \equiv 0$ and $F_\theta(X_{n+1:n}) \equiv 1$.

In the consequence of the distribution function F_θ and random sample properties the specified variables D_i are independent and identically distributed with following mean values:

$$E\left(D_i\right) = \frac{1}{n+1}, \quad i = 1, \ldots, n.$$

The discussed method is based on a fact that for a sufficiently large sample of observed values the sequence of the $\{D_i(\theta)\}_{i=1,\ldots,n}$ is asymptotically equal to the sequence of their theoretical mean values $\left\{\frac{1}{n+1}\right\}_{i=1,\ldots,n}$. To measure the distance between these sequences in the GSE method the Csiszar h-divergence is proposed. In terms of specified concepts the GSEs are found by minimizing the following expression with respect to θ:

$$-\sum_{i=1}^{n} \frac{h\left((n+1)\cdot D_i(\theta)\right)}{n+1},$$

where h is a strictly convex, real-valued function defined on the interval $(0, \infty)$ and such that $h(1) = 0$.

The choice of the h function is not unique, therefore the GSE is not one estimator, but rather a category of estimators with different properties. The first proposed and the most common-used estimator across GSEs class is the maximum product of spacings (MPS) estimator suggested by Cheng and Amin [4] for $h(x) = -log(x)$. The h-divergence specified in this case is the well-known Kullback-Leibler divergence widely applied in the entropy theory. The MPS estimator is found by maximizing the following equation:

$$\sum_{i=1}^{n} \ln D_i(\theta) = \sum_{i=1}^{n} \log \left(F_\theta \left(X_{i:n}\right) - F_\theta \left(X_{i-1:n}\right)\right).$$

In [9] and [2] there was proved that the MPS estimator has the best theoretical properties among estimators provided by the generalize spacing method. Therefore only the MPS method was included in the simulation study. See also other papers [7, 8] for more details about properties of GSEs (consistency, asymptotic normality) and their applications in practice.

3.6. The percentile estimation method

The mathematical form of the Weibull distribution allows application of one more estimation method. The proposed technique is based on percentiles and is structurally similar to the traditional method of moments.

For the Weibull distribution and percentile x_p specified for percent p the following equation is true:

$$p = 1 - \exp\left(-\left(\frac{x_p}{\lambda}\right)^k\right).$$

Based on the above equation determined for two selected and different percentiles $\hat{x}_1$ and $\hat{x}_2$ corresponding to p_1 and p_2 percents, respectively, we get the following percentile method (PE) estimates of the unknown parameters:

$$\hat{k} = \frac{\ln\left(\ln\left(\frac{1}{1-p_1}\right)\right) - \ln\left(\ln\left(\frac{1}{1-p_2}\right)\right)}{\ln\left(\hat{x}_1 - \hat{x}_2\right)}, \quad \hat{\lambda} = \hat{x}_1 \cdot \exp\left(-\frac{1}{\hat{k}} \cdot \ln\left(\ln\left(\frac{1}{1-p_1}\right)\right)\right).$$

In the conducted simulation study the 25% and 75% percentiles in the PE simulation were considered.

4. The simulation study

This section contains the description and results of simulation-based comparisons of presented estimators in terms of their bias and efficiency. Mean values, confidence intervals and mean square errors (MSE) of estimators were calculated over 10,000 replications generated for each of the discussed probability distribution with different values of distributions parameters and sample sizes ($n = 5, 20, 100$) included in the study. The scale and shape parameter values were taken from published studies of the mice, rats and humans survival, road traffic and food storage failures or the DNA sequences distribution [11, 13, 15, 16, 17]. The included data represent the main cases of survival curve shapes and scales observed in practice.

The comparison of considered methods was based on the MSE quantity defined as:

$$MSE(\{\hat{\lambda}_i\}_i, \{\hat{k}_i\}_i, \lambda, k) = \sqrt{\frac{1}{N} \cdot \sum_{i=1}^{N} \Big((\hat{\lambda}_i - \lambda)^2 + (\hat{k}_i - k)^2 \Big)},$$

where $\{\hat{\lambda}_i\}_i$ and $\{\hat{k}_i\}_i$ are sequences of estimates values for scale and shape parameters, while λ and k are the true parameter values. All computations were conducted using the R statistical program.

Lack of an explicit algebraic solution of the estimation problem was often the case, therefore numerical calculations were required. In the conducted simulations the Newton-Raphson's (in univariate cases) and the Nelder-Mead's (in multivariate cases) methods were applied. These two algorithms are implemented in the R program as UNIROOT and OPTIM procedures, respectively. It should be emphasized that in most cases in this study the interval of $(0.1, 100)$ was taken to search for the root using the UNIROOT algorithm. The only exceptions are the simulations conducted for the LSE estimates of the Gompertz distribution parameters. In that cases applications of the considered method with a standard approach (e.g. with interval $(0.1, 10)$) generates very bad results, therefore the smaller interval $(0.000001, 10)$ was applied. Necessity of applying this correction is a limitation of the LSE method.

The means of the estimates and the MSE values obtained in the conducted simulations for the Weibull, gamma and Gompertz distributions are presented in Tabs 7–9 and additionally, in a graphical way in Figs 1–3. For the Weibull and gamma distributions numerical calculations were convergent, while in some replications for the Gompertz distribution discussed estimation methods failed. To present convergence of estimation methods for this distribution, in Tab. 9 the numbers of convergent replications (denoted by N) in the conducted simulations were additionally given. The bold font inside Tabs 7, 8 and 9 indicates the best estimator in a given estimation problem.

Tab. 7. Simulation results for the Weibull distribution

	n = 5			n = 20			n = 100		
Method	$\hat{k}$	$\hat{\lambda}$	MSE	$\hat{k}$	$\hat{\lambda}$	MSE	$\hat{k}$	$\hat{\lambda}$	MSE
$k = 0.60$, $\lambda = 3.71$									
MM	1.056	5.231	4.281	0.738	4.334	1.842	0.639	3.919	0.813
MLE	0.872	4.329	3.489	**0.647**	**3.878**	**1.527**	**0.609**	**3.756**	**0.663**
LME	**0.705**	**3.420**	**3.162**	0.628	3.623	1.529	0.607	3.713	0.685
PE	1.463	5.269	4.779	0.799	4.906	2.888	0.642	3.952	1.235
LSE1	0.539	5.550	4.767	0.537	4.271	1.861	0.571	3.892	0.732
LSE2	0.630	4.981	4.133	0.577	4.171	1.760	0.586	3.837	0.716
MPS	0.585	5.009	4.131	0.563	4.042	1.646	0.583	3.774	0.666
$k = 1.00$, $\lambda = 13999.96$									
MM	1.523	14828	6787.3	1.118	14308	3372.0	1.023	14045	1523.8
MLE	1.443	14467	6750.3	1.074	14198	3324.6	1.014	14013	1468.0
LME	**1.146**	**13099**	**6529.7**	1.028	13837	3311.8	1.004	13960	1483.0
PE	2.382	15569	7826.9	1.310	15433	5015.7	1.066	14373	2700.3
LSE1	0.903	16581	8140.8	0.893	15026	3808.2	0.952	14350	1583.0
LSE2	1.055	15766	7597.0	0.961	14693	3545.9	0.976	14231	1575.8
MPS	0.973	15570	7117.0	**0.939**	**14580**	**3296.2**	**0.973**	**14192**	**1423.3**
$k = 3.64$, $\lambda = 0.81$									
MM	5.195	0.800	3.313	3.901	0.806	0.825	3.687	0.809	0.302
MLE	5.254	0.799	3.285	3.931	0.808	0.816	**3.688**	**0.809**	**0.295**
LME	4.317	0.804	2.636	3.748	0.809	0.762	3.658	0.810	0.307
PE	8.398	0.805	6.679	4.631	0.815	1.443	3.865	0.812	0.513
LSE1	**3.279**	**0.831**	**1.905**	3.242	0.821	0.852	3.459	0.814	0.411
LSE2	3.829	0.818	2.158	3.490	0.816	0.819	3.559	0.813	0.383
MPS	3.534	0.818	1.942	**3.409**	**0.813**	**0.690**	3.536	0.810	0.300

Tab. 8. Simulation results for the gamma distribution

Method	n = 5 $\hat{k}$	n = 5 $\hat{\lambda}$	n = 5 MSE	n = 20 $\hat{k}$	n = 20 $\hat{\lambda}$	n = 20 MSE	n = 100 $\hat{k}$	n = 100 $\hat{\lambda}$	n = 100 MSE
k = 0.21, λ = 357142.86									
MM	**0.578**	**182591**	**282863**	0.237	343344	135023	2.009	22.675	20.109
MLE	0.376	301757	370567	**0.232**	**346170**	**196481**	**2.179**	**22.052**	**18.979**
LME	0.304	781974	1738657	0.223	420503	333249	1.545	35.905	39.289
MPS	0.229	1354910	3207240	0.205	400882	105443	1.178	52.677	54.443
k = 1.00, λ = 27.03									
MM	2.009	22.675	20.109	1.219	25.689	12.211	1.046	26.794	5.912
MLE	**2.179**	**22.052**	**18.979**	**1.144**	**25.867**	**9.556**	**1.026**	**26.722**	**4.266**
LME	1.545	35.905	39.289	1.066	28.697	12.248	1.013	27.321	4.940
MPS	1.178	52.677	54.443	0.921	34.888	15.359	0.961	29.296	5.313
k = 4.70, λ = 30.30									
MM	9.707	28.995	28.812	5.319	30.168	11.224	4.832	30.178	4.929
MLE	**10.197**	**24.657**	**22.187**	**5.503**	**28.727**	**9.946**	**4.840**	**30.003**	**4.451**
LME	7.558	35.058	29.358	5.098	31.269	11.018	4.779	30.428	4.649
MPS	5.549	53.036	47.549	4.235	38.121	14.959	4.480	32.593	5.285

Tab. 9. Simulation results for the Gompertz distribution

	n = 5				n = 20				n = 100			
Method	$\hat{k}$	$\hat{\lambda}$	MSE	N	$\hat{k}$	$\hat{\lambda}$	MSE	N	$\hat{k}$	$\hat{\lambda}$	MSE	N
$k = 0.00$, $\lambda = 0.09$												
MM	0.114	0.059	0.192	7368	0.035	0.069	0.052	5345	0.016	0.078	0.028	2556
MLE	0.118	0.059	0.201	8142	0.030	0.073	0.048	6871	0.009	0.083	0.018	5870
LME	0.109	0.060	0.188	4515	0.034	0.071	0.051	3894	0.015	0.077	0.020	2614
MPS	**-0.001**	**0.118**	**0.161**	**10000**	**-0.004**	**0.096**	**0.003**	**10000**	**-0.003**	**0.092**	**0.017**	**10000**
$k = 0.05$, $\lambda = 2.57$												
MM	2.830	1.858	4.666	8074	0.845	2.151	1.356	7010	0.308	2.395	0.495	6170
MLE	3.458	1.679	5.687	8247	0.883	2.118	1.398	7076	0.349	2.391	0.495	6536
LME	2.605	1.926	4.507	4809	0.812	2.167	1.321	5080	0.327	2.387	0.518	5301
LSE1[1]	**2.515**	**1.552**	**3.695**	**4276**	0.781	2.045	1.370	3842	0.271	2.344	0.579	3760
LSE2[1]	2.612	1.642	3.849	5243	0.902	2.052	1.568	4922	0.307	2.369	0.629	4856
MPS	0.041	3.395	4.797	10000	**-0.054**	**2.754**	**1.276**	**10000**	**-0.029**	**2.655**	**0.492**	**10000**
$k = 5.00$, $\lambda = 1.00$												
MM	6.736	1.211	3.741	8327	5.531	1.017	1.933	9967	4.897	1.001	0.779	10000
MLE	8.397	0.921	6.553	9939	5.590	0.982	1.716	10000	**5.117**	**0.943**	**0.731**	**10000**
LME	**5.823**	**1.499**	**3.499**	**8356**	5.099	1.126	1.911	9990	5.005	1.026	0.799	10000
LSE1[1]	8.226	1.156	14.310	8932	4.291	1.925	2.613	9834	5.025	1.125	0.984	9989
LSE2[1]	1.018	1.026	15.147	9292	5.267	1.110	3.638	9898	4.970	1.055	0.902	9988
MPS	4.526	1.677	4.628	10000	**4.576**	**1.212**	**1.715**	**10000**	4.825	1.061	0.733	10000

[1]The $(0.000001, 10)$ interval was taken instead of $(0.1, 100)$ to search for the root using the UNIROOT algorithm; using the standard approach the LSE method generates very bad results.

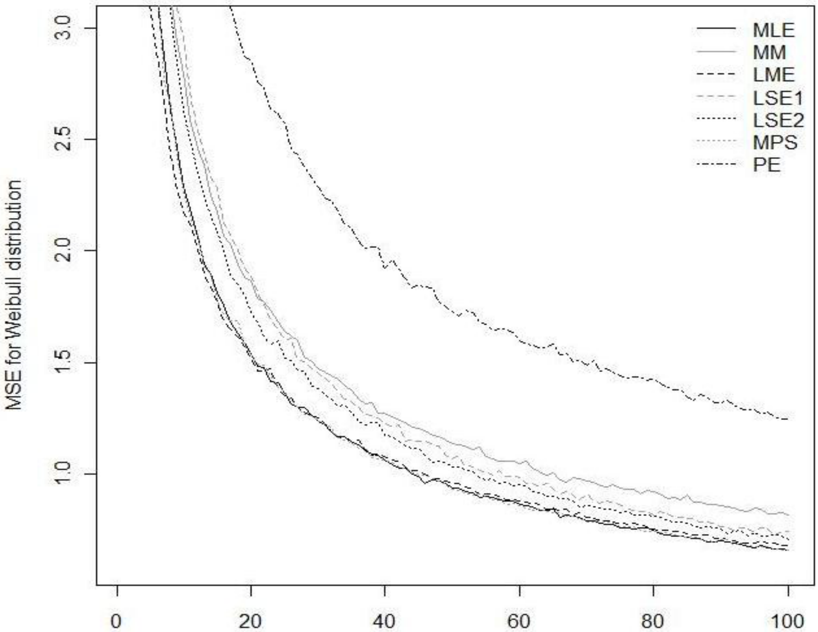

Fig. 1. An illustrative overview of the MSE values from the simulation study for the Weibull distribution with true parameters: $\lambda = 3.71$ and $k = 0.60$

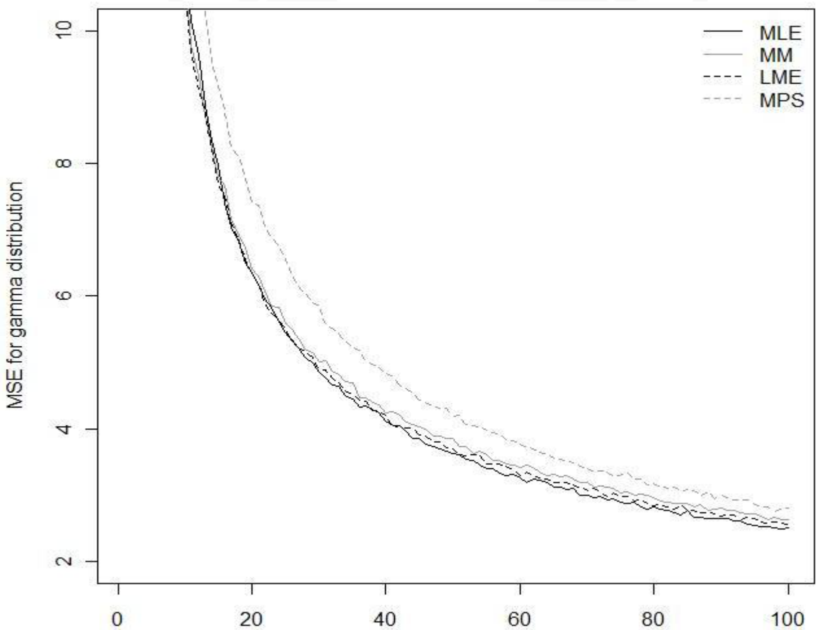

Fig. 2. An illustrative overview of the MSE values from the simulation study for the gamma distribution with true parameters: $\lambda = 30.30$ and $k = 4.70$

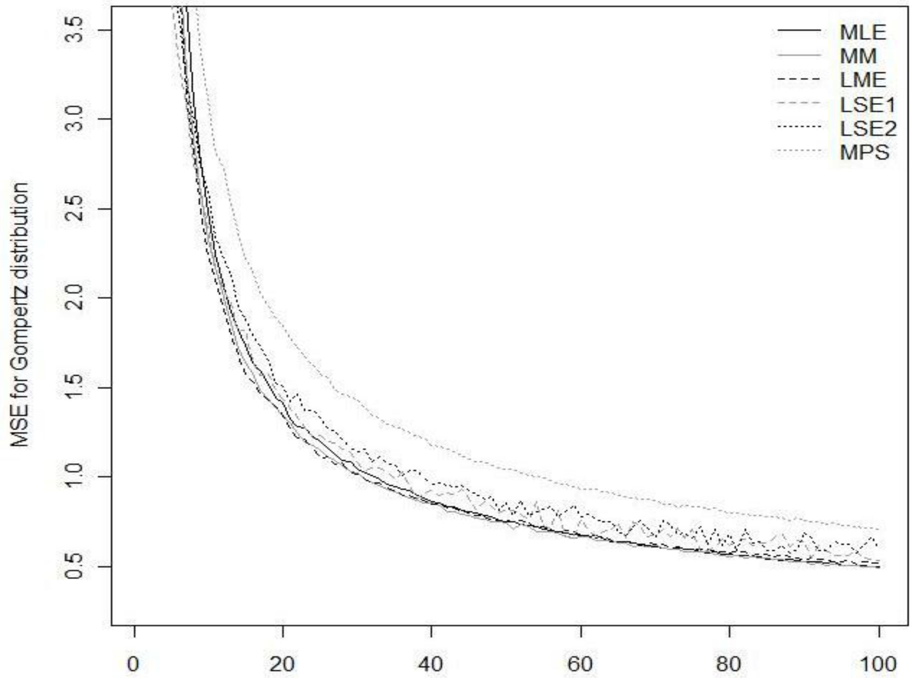

Fig. 3. An illustrative overview of the MSE values from the simulation study for the Gompertz distribution with true parameters: $\lambda = 2.57$ and $k = 0.05$

There is a broad spectrum of available estimation methods in cases of the Weibull and the Gompertz distributions. Except for the percentile method, which provides relatively poor approximation of true distribution parameters, the others of the discussed estimators generate roughly comparable results and bias.

Based on the simulation results given in Tabs 7 and 9 (generated for the Weibull and the Gompertz distributions) the MLE, LME and MPS estimators present the smallest bias in terms of the MSE, among described estimation techniques and over most of the considered parameter values and sample sizes. Therefore in the case of the Weibull and the Gompertz distributions these estimators are preferred.

For the gamma distribution only traditional methods are proposed (Tab. 8). The character of the gamma distribution function causes that results of the others discussed estimators have a large variance or their application is very problematic.

Most of the considered estimators do not have explicit algebraic formulations. Therefore estimation of the Weibull, gamma or Gompertz distribution parameters often requires application of numerical methods and selection of proper starting values for these calculations. In the conducted simulation study different initial values were considered and analyzed for their influence on the final results (data not shown). Particularly sensitive to these variations were the LSE and MPS estimators, while the traditional estimation methods (MLE and MM) and the LME were almost completely resistant to them.

Additionally, it should be emphasized that the results of the simulation study are always related with some kind of uncertainty, caused by the numerical calculation. This effect is a result of both application of the numerical methods and computer evaluation. The discussed uncertainty should be taken into consideration while interpreting results in each of the considered cases.

5. Conclusions

Overall, based on the simulation results, it is clear that we could not indicate one and the best estimation method in general. In each case the choice of the appropriate estimator should be taken carefully, particularly with consideration of the given sample size and assumed probability distribution. On the other hand, obtained results may suggest that application of the traditional, well-known methods is justified in most cases.

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7. References

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