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An abstract nonlocal functional-differential second order evolution problem

Abstrakcyjne nielokalne funkcjonalno-różniczkowe ewolucyjne zagadnienie rzędu drugiego

Abstract

The aim of the paper is to prove two theorems on the existence and uniqueness of mild and classical solutions of a semilinear functional-differential evolution second order equation together with nonlocal conditions. The theory of strongly continuous cosine families of linear operators in a Banach space is applied. The paper is based on publications [1–9] and is a generalization of paper [6].

Keywords: nonlocal, second order, functional-differential, evolution problem, Banach space

Streszczenie

W artykule udowodniono dwa twierdzenia o istnieniu i jednoznaczności całkowych i klasycznych rozwiązań semiliniowego funkcjonalno-różniczkowego zagadnienia ewolucyjnego rzędu drugiego z warunkami nielokalnymi. W tym celu zastosowano teorię rodziny cosinus liniowych operatorów w przestrzeni Banacha. Artykuł bazuje na publikacjach [1–9] i jest pewnym uogólnieniem publikacji [6].

Słowa kluczowe: nielokalne, rzędu drugiego, funkcjonalno-różniczkowe, zagadnienie ewolucyjne, przestrzeń Banacha

1. Introduction

In this paper, we consider the abstract nonlocal semilinear functional-differential second order Cauchy problem

$$u''(t) = Au(t) + f(t, u(t), u(a(t)), u'(t)),\ t \in (0, T], \tag{1}$$

$$u(0) = x_0, \tag{2}$$

$$u'(0) + \sum_{i=1}^{p} h_i u(t_i) = x_1, \tag{3}$$

where A is a linear operator from a real Banach space X into itself, $u:[0,T] \to X$, $f:[0,T] \times X^3 \to X$, $a:[0,T] \to [0,T]$, $x_0, x_1 \in X$, $h_i \in \mathbb{R}$ $(i = 1, 2, ..., p)$ and $0 < t_1 < t_2 < ... < t_p \leq T$.

We prove two theorems on the existence and uniqueness of mild and classical solutions of problem (1)–(3). For this purpose we apply the theory of strongly continuous cosine families of linear operators in a Banach space. We also apply the Banach contraction theorem and the Bochenek theorem (see Theorem 1.1 in this paper).

Let A be the same linear operator as in (1). We will need the following assumption:

Assumption (A_1). Operator A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators from X into itself.

Recall that the infinitesimal generator of a strongly continuous cosine family $C(t)$ is the operator $A : X \supset D(A) \to X$ defined by

$$Ax := \frac{d^2}{dt^2} C(t)x\Big|_{t=0},\ x \in D(A),$$

where

$$D(A) := \{x \in X :\ C(t)x \text{ is of class } C^2 \text{ with respect to } t\}.$$

Let

$$E := \{x \in X :\ C(t)x \text{ is of class } C^1 \text{ with respect to } t\}.$$

The associated sine family $\{S(t) : t \in \mathbb{R}\}$ is defined by

$$S(t)x := \int_0^t C(s)x\,ds,\ x \in X,\ t \in \mathbb{R}.$$

From Assumption (A_1) it follows (see [9]) that there are constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|C(t)\| \leq Me^{\omega|t|} \quad \text{and} \quad \|S(t)\| \leq Me^{\omega|t|} \quad \text{for } t \in \mathbb{R}.$$

We also will use the following assumption:

Assumption (A_2). The adjoint operator A^* is densely defined in X^* that is, $\overline{D(A^*)} = X^*$.

The paper is based on publications [1–9] and is a generalization of paper [6].

For convenience of the reader, a result obtained by J. Bochenek (see [2]) will be presented here. Let us consider the Cauchy problem

$$u''(t) = Au(t) + h(t), \quad t \in (0,T], \tag{4}$$

$$u(0) = x_0, \tag{5}$$

$$u'(0) = x_1. \tag{6}$$

A function $u : [0,T] \to X$ is said be a classical solution of the problem (4)–(6) if

(a) $$u \in C^1([0,T],X) \cap C^2((0,T],X),$$

(b) $$u(0) = x_0 \text{ and } u'(0) = x_1,$$

(c) $$u''(t) = Au(t) + h(t) \quad \text{for} \quad t \in (0,T].$$

Theorem 1.1. *Suppose that:*
(i) *Assumptions (A_1) and (A_2) are satisfied,*
(ii) *$h : [0,T] \to X$ is Lipschitz continuous,*
(iii) *$x_0 \in D(A)$ and $x_1 \in E$.*
Then u given by the formula

$$u(t) = C(t)x_0 + S(t)x_1 + \int_0^t S(t-s)h(s)ds, \quad t \in [0,T],$$

is the unique classical solution of the problem (4)–(6).

2. Theorem on mild solutions

A function u belonging to $C^1([0,T],X)$ and satisfying the integral equation

$$u(t) = C(t)x_0 + S(t)x_1 - S(t)\left(\sum_{i=1}^{p} h_i u(t_i)\right)$$

$$+\int_0^t S(t-s) f(s,u(s),u(a(s)),u'(s))ds, \quad t \in [0,T],$$

is said to be a mild solution of the nonlocal Cauchy problem (1)–(3).

Theorem 2.1. *Suppose that:*
(i) *Assumption (A_1) is satisfied,*
(ii) *$a : [0,T] \to [0,T]$ is of class C^1 on $[0,T]$, $f : [0,T] \times X^3 \to X$ is continuous with respect to the first variable $t \in [0,T]$ and there exists a positive constant L_1 such that*

$$\|f(s,z_1,z_2,z_3)-f(s,\tilde{z}_1,\tilde{z}_2,\tilde{z}_3)\| \le L_1\sum_{i=1}^{3}\|z_i-\tilde{z}_i\| \;\; for \;\; s\in[0,T],\, z_i,\tilde{z}_i\in X \quad (i=1,2,3),$$

(iii) $2C\left(2TL_1+\sum_{i=1}^{p}|h_i|\right)<1,$

$$where\ C:=\sup\{\|C(t)\|+\|S(t)\|+\|S'(t)\|:t\in[0,T]\},$$

(iv) $x_0\in E \;\; and \;\; x_1\in X.$

Then the nonlocal Cauchy problem (1)–(3) *has a unique mild solution.*

Proof. Let the operator $F:C^1([0,T].X)\to C^1([0,T],X)$ be given by

$$(Fu)(t)=C(t)x_0+S(t)x_1-S(t)\left(\sum_{i=1}^{p}h_iu(t_i)\right)$$

$$+\int_0^t S(t-s)f(s,u(s),u(a(s)),u'(s))ds,\;\; t\in[0,T].$$

Now, we shall show that F is a contraction on the Banach space $C^1([0,T],X)$ equipped with the norm

$$\|w\|_1:=\sup\{\|w(t)\|+\|w'(t)\|:t\in[0,T]\}.$$

To do this, observe that

$$\|(Fw)(t)-(F\tilde{w})(t)\|=\Bigg\|S(t)\left(\sum_{i=1}^{p}h_i(\tilde{w}(t_i)-w(t_i))\right)$$

$$+\int_0^t S(t-s)(f(s,w(s),w(a(s)),w'(s))-f(s,\tilde{w}(s),\tilde{w}(a(s)),\tilde{w}'(s)))ds\Bigg\|$$

$$\le C\left(\sum_{i=1}^{p}|h_i|\right)\|w-\tilde{w}\|_1$$

$$+\int_0^t\|S(t-s)\|L_1\left(\|w(s)-\tilde{w}(s)\|+\|w(a(s))-\tilde{w}(a(s))\|+\|w'(s)-\tilde{w}'(s)\|\right)ds$$

$$\le C\left(2TL_1+\sum_{i=1}^{p}|h_i|\right)\|w-\tilde{w}\|_1$$

and

$$\|(Fw)'(t)-(F\tilde{w})'(t)\|=\Bigg\|S'(t)\left(\sum_{i=1}^{p}h_i(\tilde{w}(t_i)-w(t_i))\right)$$

$$+\int_0^t C(t-s)\big(f(s,w(s),w(a(s)),w'(s))-f(s,\tilde{w}(s),\tilde{w}(a(s)),\tilde{w}'(s))\big)ds\Bigg\|$$

$$\le C\left(\sum_{i=1}^{p}|h_i|\right)\|w-\tilde{w}\|_1$$

$$+\int_0^t \|C(t-s)\|L_1\left(\|w(s)-\tilde{w}(s)\|+\|w(a(s))-\tilde{w}(a(s))\|+\|w'(s)-\tilde{w}'(s)\|\right)ds$$

$$\le C\left(2TL_1+\sum_{i=1}^{p}|h_i|\right)\|w-\tilde{w}\|_1,\quad t\in[0,T].$$

Consequently

$$\|Fw-F\tilde{w}\|_1\le 2C\left(2TL_1+\sum_{i=1}^{p}|h_i|\right)\|w-\tilde{w}\|_1\quad \text{for}\quad w,\tilde{w}\in C^1([0,T],X).$$

Therefore, in space $C^1([0,T],X)$ there is the only one fixed point of F and this point is the mild solution of the nonlocal Cauchy problem (1)–(3). So, the proof of Theorem 2.1 is complete.

Remark 2.1. The application of a Bielecki norm in the proof of Theorem 2.1 does not give any benefit.

3. Theorem about classical solutions

A function $u:[0,T]\to X$ is said to be a classical solution to the problem (1)–(3) if

(a) $$u\in C^1([0,T],X)\cap C^2(([0,T],X),$$

(b) $$u(0)=x_0\quad\text{and}\quad u'(0)+\sum_{i=1}^{p}h_i u(t_i)=x_1,$$

(c) $$u''(t)=Au(t)+f(t,u(t),u(a(t)),u'(t))\quad\text{for}\quad t\in[0,T].$$

Theorem 3.1. *Suppose that:*

(i) *Assumptions* (A_1) *and* (A_2) *are satisfied, and* $a:[0,T]\to[0,T]$ *is of class* C^1 on $[0,T]$.

(ii) *There exists a positive constant* L_2 *such that*

$$\|f(s,z_1,z_2,z_3)-f(\tilde{s},\tilde{z}_1,\tilde{z}_2,\tilde{z}_3)\|\le L_2\left(|s-\tilde{s}|+\sum_{i=1}^{3}\|z_i-\tilde{z}_i\|\right)$$

for $s,\tilde{s}\in[0,T],\ z_i,\tilde{z}_i\in X\quad(i=1,2,3)$.

(iii) $2C\left(2TL_2+\sum_{i=1}^{p}|h_i|\right)<1.$

(iv) $x_0 \in E$ and $x_1 \in X$.

Then the nonlocal Cauchy problem (1)–(3) *has a unique mild solution u. Moreover, if*

$$x_0 \in D(A), \quad x_1 \in E \quad and \quad u(t_i) \in E \quad (i=1,2,...,p),$$

and if there exists a positive constant κ *such that*

$$\|u(a(s))-u(a(\tilde{s}))\| \le \kappa \|u(s)-u(\tilde{s})\| \quad for \quad s,\tilde{s} \in [0,T]$$

then u is the unique classical solution of nonlocal problem (1)–(3).

Proof. Since the assumptions of Theorem 2.1 are satisfied, the nonlocal Cauchy problem (1)–(3) possesses a unique mild solution which is denoted by *u*.

Now, we shall show that *u* is the classical solution of problem (1)–(3).

Firstly, we shall prove that *u*, $u(a(\cdot))$and $u`$ satisfy the Lipschitz condition on $[0,T]$. Let t and $t+h$ be any two points belonging to $[0,T]$. Observe that

$$u(t+h)-u(t)=C(t+h)x_0+S(t+h)x_1-S(t+h)\left(\sum_{i=1}^{p} h_i u(t_i)\right)$$

$$+\int_0^{t+h} S(t+h-s)f(s,u(s),u(a(s)),u'(s))ds$$

$$-C(t)x_0-S(t)x_1+S(t)\left(\sum_{i=1}^{p} h_i u(t_i)\right)$$

$$-\int_0^{t} S(t-s)f(s,u(s),u(a(s)),u'(s))ds.$$

Since

$$C(t)x_0+S(t)\left(x_1-\sum_{i=1}^{p} h_i u(t_i)\right)$$

is of class C^2 in $[0,T]$, there are $C_1>0$ and $C_2>0$ such that

$$\left\|\left(C(t+h)-C(t)\right)x_0+\left(S(t+h)-S(t)\right)\left(x_1-\sum_{i=1}^{p} h_i u(t_i)\right)\right\| \le C_1|h|$$

and

$$\left\|\left((C(t+h)-C(t))x_0\right)'+\left((S(t+h)-S(t))(x_1-\sum_{i=1}^{p} h_i u(t_i)\right)'\right\| \le C_2|h|.$$

Hence

$$\|u(t+h)-u(t)\| \le C_1|h| + \left\| \int_0^t S(s)\big(f(t+h-s,u(t+h-s),u(a(t+h-s)),u'(t+h-s))\right.$$

$$\left. - f(t-s,u(t-s),u(a(t-s)),u'(t-s))\big)ds \right\|$$

$$+ \left\| \int_t^{t+h} S(s) f(t+h-s,u(t+h-s),u(a(t+h-s),u'(t+h-s))ds \right\|$$

$$\le C_1|h| + \int_0^t Me^{\omega T} L_2 \big(|h| + \|u(t+h-s)-u(t-s)\| + \|u(a(t+h-s))-u(a(t-s))\|$$

$$+ \|u'(t+h-s)-u'(t-s)\|\big)ds + Me^{\omega T} N|h|,$$

where

$$N := \sup\{\|f(s,u(s),u(a(s)),u'(s))\| : s \in [0,T]\}.$$

From this we obtain

$$\|u(t+h)-u(t)\| \le C_3|h| + C_4 \int_0^t \big(\|u(s+h)-u(s)\| + \|u'(s+h)-u'(s)\|\big)ds. \qquad (7)$$

Moreover, we have

$$u'(t) = \left(C(t)x_0 + S(t)\left(x_1 - \sum_{i=1}^{p} h_i u(t_i) \right) \right)' + \int_0^t C(t-s) f(s,u(s),u(a(s)),u'(s))ds.$$

From the above formula we obtain, analogously

$$\|u'(t+h)-u'(t)\| \le C_5|h| + C_6 \int_0^t \big(\|u(s+h)-u(s)\| + \|u'(s+h)-u'(s)\|\big)ds. \qquad (8)$$

By inequalities (7) and (8), we get

$$\|u(t+h)-u(t)\| + \|u'(t+h)-u'(t)\|$$

$$\le C_*|h| + C_{**} \int_0^t \big(\|u(s+h)-u(s)\| + \|u'(s+h)-u'(s)\|\big)ds.$$

From Gronwall's inequality, we have

$$\|u(t+h)-u(t)\| + \|u'(t+h)-u'(t)\| \le \tilde{C}|h|, \qquad (9)$$

where $\tilde{C}$ is a positive constant.

By (9), it follows that u, $u(a(\cdot))$ and u' satisfy the Lipschitz condition on $[0,T]$ with a positive constant $\tilde{C}$. This implies that the mapping

$$[0,T] \ni t \to f(t,u(t),u(a(t)),u'(t)) \in X$$

also satisfies the Lipschitz condition.

The above property of *f* together with the assumptions of Theorem 3.1 imply, by Theorem 1.1 and by Theorem 2.1, that the linear Cauchy problem

$$v''(t) = Av(t) + f(t,u(t),u(a(t)),u'(t)),\ \ t \in [0,T],$$

$$v(0) = x_0,$$

$$v'(0) = x_1 - \sum_{i=1}^{p} h_i u(t_i)$$

has a unique classical solution *v* such that

$$v(t) = C(t)x_0 + S(t)\left(x_1 - \sum_{i=1}^{p} h_i u(t_i) \right)$$

$$+ \int_0^t S(t-s) f(s,u(s),u(a(s)),u'(s))ds = u(t),\ \ \ t \in [0,T].$$

Consequently, *u* is the unique classical solution of the semilinear Cauchy problem (1)–(3) and, therefore, the proof of Theorem 3.1 is complete.

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