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The p-Factor Method for Nonlinear Optimization

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Abstract. We present the main concept and results of the p-regularity theory (also known as p-factor analysis of nonlinear mappings) applied to nonlinear optimization problems. The approach is based on the construction of p-factor operator. The main result of this theory gives a detailed description of the structure of the zero set of irregular nonlinear mappings. Applications include a new numerical method for solving nonlinear optimization problems and p-order necessary and sufficient optimality conditions. We substantiate the rate of convergence of p-factor method.

Keywords: nonlinear optimization, p-factor operator, p-regularity, kernel, singularity, convergence, necessary and sufficient conditions.

1. Introduction

Consider the following nonlinear constrained optimization problem

$$\min \phi(x), \tag{1}$$

subject to

$$F(x) = 0, \tag{2}$$

where $F : \mathbb{R}^n \to \mathbb{R}^m$ and $F(x) = (f_1(x), f_2(x), \cdots, f_m(x))^T$, $\phi : \mathbb{R}^n \to \mathbb{R}$, $F \in \mathcal{C}^{p+1}(\mathbb{R}^n)$, $p \in \mathbb{N}$, $\phi \in \mathcal{C}^2(\mathbb{R}^n)$ and at the solution x^* we have $\mathrm{Im}F'(x^*) \neq \mathbb{R}^m$.

The problem (1)–(2) is said to be *regular* at the solution x^* if the Jacobian matrix has a full rank, i.e.

$$\operatorname{rank} F'(x^*) = m. \tag{3}$$

In the case, when the Jacobian matrix $F'(x^*)$ is singular, the problem (1)–(2) is said to be *irregular* (*degenerate*) at x^*, and the point x^* is said to be a *singular* (*degenerate*) solution to the problem. Most results on convergence properties of methods for solving optimization problems presuppose the regularity assumption (3). If the regularity assumption does not hold, these methods lose their high convergence rate or simply become inapplicable for finding solutions.

Let us note that it was proved in [6] that essentially nonlinear problems (that is, problems which cannot be equivalently reformulated as locally linear problems) are equivalent to degenerate problems. Our aim is to describe some degenerate optimization problems.

For a better illustration of the results derived in the paper, first we apply a classical result to the degenerate optimization problem.

In the classical case, for the equation $F(x) = 0$ where the operator $F'(x^*)$ is non-singular, the tangent cone $T_1M(x^*)$ to the set $M(x^*) = \{x \in U_\varepsilon(x^*) \mid F(x) = F(x^*)\}$ at x^* is equal to the kernel of $F'(x^*)$ (Lyusternik theorem). But in the degenerate case, Lyusternik theorem is inapplicable for the description of the solution set. If the operator $F'(x^*)$ is singular then it is possible that $T_1M(x^*) \neq \mathrm{Ker}F'(x^*)$. For example, if F is given by $F(x) = x_1^2 - x_2^2 + o(\|x\|^2)$ and $x^* = 0$, then $F'(x^*) = 0$ and $\mathrm{Ker}F'(0) = 0$. Hence $\mathrm{Ker}F'(0) \neq \mathbb{R}^2 = T_1M(x^*) = \left\{ \begin{pmatrix} t \\ t \end{pmatrix} \cup \begin{pmatrix} t \\ -t \end{pmatrix}, t \in \mathbb{R} \right\}$.

We observe the same phenomenon when we consider optimality conditions. In the regular case we can apply the Lagrange theorem, which says that, if x^* is the minimum of $\phi(x)$ and $F'(x^*)\mathbb{R}^n = \mathbb{R}^m$ then there exists $\lambda^* \in \mathbb{R}^m$ such that $\phi'(x^*) = F'(x^*)^T\lambda^*$. But in the irregular case the Lagrange theorem may fail. For example let us consider the problem of minimizing $\phi(x) = x_1^2 + x_3$, subject to $F(x) = \begin{pmatrix} x_1^2 - x_2^2 + x_3^2 \\ x_1^2 - x_2^2 + x_3^2 + x_2x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, where $x^* = (0,0,0)^T$ is a solution point. Here $\phi'(x^*) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $F'(x^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Hence, we can note that $\phi'(0) \neq F'(0)^T\lambda$.

Moreover, when we consider the numerical solution of the nonlinear equation of the form (2) in the singular case, the properties of numerical methods may fail.

Example 1. Consider the following mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x) = \begin{pmatrix} x_1 + x_2 \\ x_1x_2 \end{pmatrix},$$

where $x^* = (0,0)^T$ is the solution of the equation $F(x) = 0$ and $F'(x^*) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is singular at x^*.

Let us use the classical Newton method for solving this problem:

$$x_{k+1} = x_k - \{F'(x_k)\}^{-1}F(x_k), \quad k = 0, 1, 2, 3, \ldots$$

and let $x_0 = (x_1^0, x_2^0)^T$, $x_0 \in U_\varepsilon(0)$ ($\varepsilon > 0$ is sufficiently small number). Then for

$k = 1$,

$$x_1 = \frac{1}{x_1^0 - x_2^0} \begin{pmatrix} -x_1^0 x_2^0 \\ x_1^0 x_2^0 \end{pmatrix}.$$

For $x_1^0 = x_2^0$, the inverse matrix $\{F'(x_0)\}^{-1}$ does not exist and the Newton method is inapplicable. But even though $\{F'(x_0)\}^{-1}$ exists for some x_0, e.g. for $x_0 = (t + t^3, t)^T$, we have $x_1 = \begin{pmatrix} -\frac{1}{t} - t \\ \frac{1}{t} + t \end{pmatrix}$ and $\|x_1 - 0\| \approx \frac{1}{t} \to \infty$, when $t \to 0$. If $t = 10^{-5}$ then $\|x_1 - 0\| \approx 10^5$. Rejecting effect!

In the case when we would like to use the Newton method for unconditional optimization problems we also do not obtain satisfactory results. For example, we consider the following problem

$$\min_{x \in \mathbb{R}^2} \phi(x)$$

$$x_{k+1} = x_k - \{\phi''(x_k)\}^{-1} \phi'(x_k),$$

where $\phi(x) = x_1^2 + x_1^2 x_2 + x_2^4$, and $x^* = (0, 0)^T$.
At the initial point $x_0 = (x_{01}, x_{02})^T$, where $x_{01} = x_{02}\sqrt{6(1 + x_{02})}$, we have

$$\phi''(x_0) = \begin{pmatrix} 2 + 2x_{02} & 2x_{02}\sqrt{6(1 + x_{02})} \\ 2x_{02}\sqrt{6(1 + x_{02})} & 12x_{02}^2 \end{pmatrix},$$

and $\det \phi''(x_0) = 0$, hence the inverse matrix $\{\phi''(x_0)\}^{-1}$ does not exist.

It shows that if the problem is irregular we can not use the classical methods for solving it. We would like to show how to apply the so called p-regularity theory to solve irregular problems. The construction of p-regularity introduced in [7, 8] gives new possibilities for description and investigation of the irregular solutions of degenerate nonlinear optimization problems. One of the applications is that the construction of p-factor-operator is used to create numerical methods for solving degenerate optimization problems and to state p-order necessary and sufficient optimality conditions.

2. Elements of p-regularity theory

Let $F \in C^{p+1}(\mathbb{R}^n)$. For k-th order derivative $F^{(k)}(x^*)$ at $x^* \in \mathbb{R}^n$, the associated k-form is denoted by

$$F^{(k)}(x^*)[h]^k = F^{(k)}(x^*)[h, \ldots, h].$$

Suppose that the space Y is decomposed into a direct sum

$$Y = Y_1 \oplus \cdots \oplus Y_p, \tag{4}$$

where $Y_1 = \text{Im} F'(x^*),\ Z_1 = Y$. Let Z_2 be closed complementary subspace to Y_1 (we assume that such closed complement exists), and let $P_{Z_2} : Y \to Z_2$ be the projection operator onto Z_2 along Y_1. By Y_2 we mean the closed linear span of the image of the quadratic map $P_{Z_2}F^{(2)}(x^*)[\cdot]^2$. More generally, define inductively,

$$Y_i = \overline{\text{span Im}P_{Z_i}F^{(i)}(x^*)[\cdot]^i} \subseteq Z_i, \;\; i = 2, \ldots, p-1,$$

where Z_i is a chosen closed complementary subspace for $Y_1 \oplus \cdots \oplus Y_{i-1}$ with respect to Y, $i = 2, \ldots, p$ and $P_{Z_i} : Y \to Z_i$ is the projection operator onto Z_i along $Y_1 \oplus \cdots \oplus Y_{i-1}$ with respect to Y, $i = 2, \ldots, p$. Finally, $Y_p = Z_p$. The order p is chosen as the minimal number for which (4) holds. Let us define the following mappings

$$F_i(x) = P_{Y_i}F(x), \quad F_i : X \to Y_i \;\; i = 1, \ldots, p,$$

where $P_{Y_i} : Y \to Y_i$ is the projection operator onto Y_i along $Y_1 \oplus \cdots \oplus Y_{i-1} \oplus Y_{i+1} \oplus \cdots \oplus Y_p$ with respect to Y, $i = 1, \ldots, p$.

Definition 1. *The linear operator $\Psi_p(h) \in \mathcal{L}(X, Y_1 \oplus \cdots \oplus Y_p)$, $h \in X$, $h \neq 0$ defined by*

$$\Psi_p(h) = F_1'(x^*) + F_2''(x^*)h + \cdots + F_p^{(p)}(x^*)[h]^{p-1},$$

is called the p-factor operator at the point x^*.

Definition 2. *We say that the mapping F is p-regular at x^* along h, if*

$$\text{Im}\Psi_p(h) = Y.$$

Definition 3. *We say that the mapping F is p-regular at x^* if it is p-regular along any h from the set*

$$H_p(x^*) = \bigcap_{k=1}^{p} \text{Ker}^k F_k^{(k)}(x^*) \setminus \{\mathbf{0}\},$$

where

$$\text{Ker}^k F_k^{(k)}(x^*) = \{\xi \in X : F_k^{(k)}(x^*)[\xi]^k = 0\}$$

is the k-kernel of the k-order mapping $F_k^{(k)}(x^*)[\cdot]^k$.

Definition 4. *We say the mapping F satisfies the normal condition of p-regularity at x^* with respect to $\overline{h} \in X$ such that $\|\overline{h}\| = 1$, if F is p-regular at x^* along $h = P_{H_p(x^*)}\,\overline{h}$, where $h \neq 0$.*

Lemma 1. *Let the mapping F be p-regular at x^* and let* $\text{rank}\, F'(x^*) < m$. *Then $H_p(x^*) = \{0\}$ if and only if there exists a neighborhood $U(x^*)$ such that the set*

$$D = \{x \in U(x^*) \mid F(x) = 0\}$$

consists of the unique element x^*.

The proof of this lemma follows immediately from the generalization of the Lyusternik theorem (see [8]), which asserts that the tangent cone $T_1M(x^*)$ to the set $M(x^*) = \{x \in U \mid F(x) = F(x^*)\}$ at the point x^* coincides with $H_p(x^*)$, i.e.,

$$T_1M(x^*) = H_p(x^*).$$

REMARK 1. By virtue of Lemma 1, we consider the case $H_p(x^*) \neq \{0\}$, because in the other case the feasible set includes only one element.
The following result (see [2]) gives the necessary optimality conditions for degenerate nonlinear optimization problems with equality constraints.

THEOREM 1 (NECESSARY OPTIMALITY CONDITIONS) (see [5]) *Let $x^* \in \mathbb{R}^n$ be a local minimizer to (1)-(2), $\phi : U \to \mathbb{R}$ be twice continuously differentiable at x^*, and $F \in C^{p+1}(\mathbb{R}^n)$. Assume that F is p-regular at x^* along an element $h^* \in H_p(x^*)$. Then there exists $\lambda^* \in \mathbb{R}^m$ such that*

$$\phi'(x^*)+(F'(x^*)+P_1F''(x^*)[h^*]+P_2F'''(x^*)[h^*]^2+...+P_{p-1}F^{(p)}(x^*)[h^*]^{p-1})^T\,\lambda^* = 0.$$

In the following theorem, we state the sufficient conditions for the optimality. Introduce the *p-factor Lagrange function* as

$$L_p(x,h,\lambda) = \varphi(x) + \langle (P_1F(x) + P_2F'(x)[h] + ... + P_pF^{(p-1)}(x)[h]^{p-1}), \lambda\rangle$$

and

$$\overline{L}_p(x,h,\lambda) = \varphi(x) + \langle (P_1F(x) + \frac{1}{3}P_2F'(x)[h] + ... + \frac{2}{p(p+1)}P_pF^{(p-1)}(x)[h]^{p-1}), \lambda\rangle$$

THEOREM 2 (SUFFICIENT OPTIMALITY CONDITIONS). *Let $\phi \in C^2(\mathbb{R}^n)$, and $F \in C^{p+1}(\mathbb{R}^n, \mathbb{R}^m)$. Assume that F is p-regular at x^* along elements $h \in H_p(x^*)$.*
If at the point x^* there exists $\gamma > 0$ and multipliers $\lambda^*(h)$ such that for all $h \in H_p(x^*)$

$$L'_{px}(x^*, h, \lambda^*(h)) = 0,$$

and

$$\langle \overline{L}''_{pxx}(x^*, h, \lambda^*(h))h, h\rangle \geq \gamma \|h\|^2$$

then the point x^* is a local minimizer to the problem (1)–(2).

3. Unconditional optimization. P-factor method

Throughout this section we consider optimization problems without constraints. Basing on the p-factor operator construction we propose a new method for solving the nonlinear equations

$$F(x) = 0, \quad F : \mathbb{R}^n \to \mathbb{R}^n$$

where the matrix $F'(x^*)$ is singular at the solution point x^*.
Let $Y_1 = \mathrm{Im} F'(x^*)$, $\bar{P}_1 = P_{Y_1^{\perp}}$,
$Y_2 = \mathrm{Im}(F'(x^*) + \bar{P}_1 F''(x^*)h)$, $\bar{P}_2 = P_{Y_2^{\perp}}$,

$\vdots$

$$Y_{k+1} = \mathrm{Im}(F'(x^*) + \sum_{i=1}^{k} \bar{P}_i F''(x^*)h + \sum_{\substack{i_2 > i_1 \\ i_1, i_2 \in \{\overline{1,k}\}}} \bar{P}_{i_2}\bar{P}_{i_1} F^{(3)}(x^*)[h]^2 + \cdots$$
$$+ \sum_{\substack{i_k > \ldots > i_1 \\ i_1,\ldots,i_k \in \{\overline{1,k}\}}} \bar{P}_{i_k} \ldots \bar{P}_{i_1} F^{(k)}(x^*)[h]^{(k-1)}), \ \ \bar{P}_{k+1} = P_{Y_{k+1}^{\perp}}, \ k = 2, \ldots, p-1.$$

Then the principal scheme of p-factor method is the following

$$x_{k+1} = x_k - \{F'(x_k) + P_1 F''(x_k)h + \cdots + P_{p-1} F^{(p)}(x_k)h^{p-1}\}^{-1} \cdot$$
$$\cdot (F(x_k) + P_1 F'(x_k)h + \cdots + P_{p-1} F^{(p-1)}(x_k)h^{p-1}), \tag{5}$$

where $P_1 = \sum_{i=1}^{p-1} \bar{P}_i$,

$P_2 = \sum_{\substack{i_2 > i_1 \\ i_1, i_2 \in \{\overline{1,p-1}\}}} \bar{P}_{i_2}\bar{P}_{i_1}$,

$\vdots$

$P_{k+1} = \sum_{\substack{i_k > \ldots > i_1 \\ i_1,\ldots,i_k \in \{\overline{1,p-1}\}}} \bar{P}_{i_k} \ldots \bar{P}_{i_1}$, $k = 2, \ldots, p-1$, h is some fixed element, $\|h\| = 1$ and $P_i, i = 1, \ldots, p-1$ are matrices of orthoprojection such that in the solution point x^* we have

$$F(x^*) + P_1 F'(x^*)h + \cdots + P_{p-1} F^{(p-1)}(x^*)h^{p-1} = 0 \tag{6}$$

and the p-factor matrix

$$F'(x^*) + P_1 F''(x^*)h + \cdots + P_{p-1} F^{(p)}(x^*)h^{p-1} \tag{7}$$

is nonsingular (p-regular along h).

It means that $\bar{P}_p = 0$, $Y_p = \mathbb{R}^n$.
Consider the case $p = 2$ for our example 1

$$x_{k+1} = x_k - \{F'(x_k) + P_1 F''(x_k)h\}^{-1} \cdot (F(x_k) + P_1 F'(x_k)h), \tag{8}$$

where P_1 is an ortoprojection onto $\mathrm{Im}(F'(x^*))^{\perp}$ and element h ($\|h\| = 1$) is such that the 2-factor matrix

$$F'(x^*) + P_1 F''(x^*)h \tag{9}$$

is nonsingular at the solution point $x^* = 0$ (2-regular along h).
Then the condition

$$F(0) + P_1 F'(0)h = 0$$

will hold and we have to solve the following equation

$$F(x) + P_1 F'(x)h = 0, \tag{10}$$

where by virtue of (9) $x^* = 0$ is a locally unique solution.

THEOREM 3. *Let $F \in C^p(\mathbb{R}^n)$ and let there exist $h, \|h\| = 1$ such that the p-factor matrix (7) is nonsingular.*
Then for any $x_0 \in U_\varepsilon(x^*)$ ($\varepsilon > 0$ sufficiently small) for the scheme (5) the following condition will be fulfilled:

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\|^2, k = 0, 1, 2, \ldots \tag{11}$$

where $c > 0$ is a constant.

Proof. Let us note that (5) it is scheme of the classical Newton's method for solving the system of equations

$$F(x) + P_1 F'(x)h + ... + P_{p-1}F^{(p-1)}(x)h^{p-1} = 0, \tag{12}$$

and the point x^* is a solution of the system (12). But under the assumption of the p-regularity of the mapping $F(x)$, the first derivative operator of the mapping considered above evaluated at the point x^* is nondegenerate. For the scheme (5), it means that are fulfilled the classical Newton conditions on convergence and rate of convergence (see [9]) then (11) holds.

Example 1. Let us consider the following problem

$$F(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $x^* = (0, 0)^T$ and

$$F'(0) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

is singular at the point $x^* = (0, 0)^T$.
The scheme of the 2-factor method is the following

$$x_{k+1} = x_k - \{F'(x_k) + P_1 F''(x_k)h\}^{-1} \cdot (F(x_k) + P_1 F'(x_k)h), \tag{13}$$

where $P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $h = (1, -1)^T$. Then after calculation

$$F'(x_k) + P_1 F''(x_k)h = \begin{pmatrix} 1 & 1 \\ x_k^2 - 1 & x_k^1 + 1 \end{pmatrix}$$

and

$$x_{k+1} = x_k - \begin{pmatrix} 1 & 1 \\ x_k^2 - 1 & x_k^1 + 1 \end{pmatrix}^{-1} \begin{pmatrix} x_k^1 + x_k^2 \\ x_k^1 x_k^2 + x_k^2 - x_k^1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 \\ x_k^2 - 1 & x_k^1 + 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ x_k^1 x_k^2 \end{pmatrix}.$$

It means that

$$\|x_{k+1} - 0\| \leq c\|x_k - 0\|^2.$$

Example 2. Consider the following optimization problem

$$\min_{x\in\mathbb{R}^2} x_1^2 + x_1^2 x_2 + x_2^4$$

$$F(x) = \varphi'(x) = \begin{pmatrix} 2x_1 + 2x_1 x_2 \\ x_1^2 + 4x_2^3 \end{pmatrix},$$

where $x^* = (0,0)^T$, and F is 3-regular at x^* along $h = (1,1)^T$.

$$\begin{aligned} F'(0) + P_1 F''(0)h + P_2 F^{(3)}(0)[h]^2 &= \varphi''(0) + P_1\varphi^{(3)}(0)h + P_2\varphi^{(4)}(0)[h]^2 \\ &= \begin{pmatrix} 2 & -11 \\ 2 & 11 \end{pmatrix} \text{– non singular!} \end{aligned}$$

Here $\bar{P}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $\bar{P}_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, $P_1 = \bar{P}_1 + \bar{P}_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}$, $P_2 = \bar{P}_2\bar{P}_1 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$.

Consider the following 3-factor scheme

$$\begin{aligned} x_{k+1} &= x_k - \left(\varphi''(0) + P_1\varphi^{(3)}(0)[h] + P_2\varphi^{(4)}(0)[h]^2\right)^{-1} \cdot \\ &\quad \cdot \left(\varphi'(x_k) + P_1\varphi''(x_k)[h] + P_2\varphi^{(3)}(x_k)[h]^2\right). \end{aligned}$$

For the sake of simplicity and best illustration we use inverse matrix at the point 0 from which the convergence rate of the method can be derived.

Let

$$x_k = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Then $\|x_{k+1} - 0\| =$

$$= \left\| x_k - \begin{pmatrix} 2 & -11 \\ 2 & 11 \end{pmatrix}^{-1} \begin{pmatrix} 2x_1 - 11x_2 + 2x_1x_2 - 6x_2^2 \\ 2x_1 + 11x_2 + x_1^2 + 18x_2^2 + 4x_2^3 \end{pmatrix} \right\| =$$

$$= \frac{1}{44}\left\| \begin{pmatrix} 11x_1^2 + 132x_2^2 + 22x_1x_2 + 44x_2^3 \\ 2x_1^2 + 48x_2^2 - 4x_1x_2 + 8x_2^3 \end{pmatrix} \right\| \le$$

$$\le 10\|x_k - 0\|^2.$$

4. P-factor method for constrained optimization problems

In this section, we present so-called p-factor Lagrange method for solving the nonlinear optimization problems with constraints given in the form $F(x) = 0$. The method is especially intended for degenerate problems where the Jacobian matrix $F'(x^*)$ is singular at the solution x^*. As it is known, the classical method is inapplicable for the degenerate optimization problems. Therefore, the main idea of the method described in this section is to construct a new nonsingular system of equations based on p-regularity theory. Under sufficient conditions for optimality the p-factor iterative sequence constructed for solving this system converges to the solution of the original nonlinear optimization problem with quadratic rate.

Let us introduce the notation $z := \binom{x}{\lambda} \in \mathbb{R}^{n+m}$, $z^* := \binom{x^*}{\lambda^*}$ and define the auxiliary mapping $R : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ by

$$R(z) = R(x, \lambda) =$$

$$\begin{pmatrix} \phi'(x) + (F'(x) + P_1 F''(x)[h^*] + ... + P_{p-1} F^{(p)}(x)[h^*]^{p-1})^T \lambda \\ F(x) + P_1 F'(x)[h^*] + ... + P_{p-1} F^{(p-1)}(x)[h^*]^{p-1} \end{pmatrix}. \tag{14}$$

Then we obtain so-called *p-factor Lagrange system of equations*

$$R(z) = R(x, \lambda) = 0, \tag{15}$$

where h^* is defined in Definition 4, and let following inequality $\operatorname{dist}(h, H_p(x^*)) \le \varepsilon$ hold for sufficiently small $\varepsilon > 0$.
The first equality in this system is necessary condition for the optimality at the point x^* (see Theorem 1) while the second raw is the new mapping which is regular at the point x^*, and $(x^*, \lambda^*)^T$ is a solution of this system.

THEOREM 4. *Let $x^* \in \mathbb{R}^n$ be a local minimizer to (1)-(2), $\phi : U \to \mathbb{R}$ a mapping twice continuously differentiable at x^* and let $F : \mathbb{R}^n \to \mathbb{R}^m$ be $(p+1)$-times continuously differentiable at $U_\varepsilon(x^*)$. Moreover assume that F is p-regular at x^* with respect to $\bar{h}$ and $R'(z^*)$ is nonsingular. Then, the point $z^* = \binom{x^*}{\lambda^*}$ is a regular local isolated solution of $R(z) = 0$.*

Proof. Recall that the system (15) is a system of $(n + m)$ equations in $(n + m)$ unknowns, and that the point z^* is its solution. For the system (14) and under assumptions of Theorem 4 the matrix $R'(z^*)$ is nonsingular. Hence (see [9]),
$z^* = (x^*, \lambda^*)^T$ is a regular local solution to (1).

To solve the system (15) we use the *p-factor Lagrange method*, which is based on the Newton‘s method:

$$\begin{pmatrix} x_{k+1} \\ \lambda_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ \lambda_k \end{pmatrix} - \{R'(x_k, \lambda_k)\}^{-1} \cdot$$

$$\cdot \begin{pmatrix} \phi'(x_k) + (F'(x_k) + P_1 F''(x_k)[h^*] + \cdots + P_{p-1} F^{(p)}(x_k)[h^*]^{p-1})^T \lambda_k \\ F(x_k) + P_1 F'(x_k)[h^*] + \cdots + P_{p-1} F^{(p-1)}(x_k)[h^*]^{p-1} \end{pmatrix}. \tag{16}$$

The next corollary follows from Theorem 4.

COROLLARY 1. *Let the assumptions of Theorem 4 be fulfilled, $x_0 \in U_\varepsilon(x^*)$ and let $\lambda_0 \in U_\varepsilon(\lambda^*)$, where $\varepsilon > 0$ is a sufficiently small number. The sequence $\{x_k\}$, generated by (16), converges to x^* quadratically, $\|x_k - x^*\| \le C q^{2^k}$, $q < 1$, $C > 0$, $k = 0, 1, 2, \ldots$.*

Proof. By assumptions that matrix $R'(z^*)$ is nonsingular (nondegenerate) at the solution point of the system (15), then the convergence conditions of Newton method are fulfilled. Taking into account the quadratic rate of convergence of Newton method and the properties of the mappings considered above, we infer that the sequence $\{z_k\} = \{(x_k,\ \lambda_k)^T\}$ converges to the point $z^* = (x^*, \lambda^*)^T$ quadratically, i.e.

$$\|z_k - z^*\| \leq C_1 q^{2^k}, \quad q < 1, \quad C_1 > 0, \quad k = 0, 1, 2, \ldots,$$

where $q = \gamma\|z^0 - z^*\|$, $\gamma > 0$. By virtue of

$$\|x_k - x^*\| \leq \|x_k - x^*\| + \|\lambda_k - \lambda^*\| \leq 2\|z_k - z^*\|, \quad k = 0, 1, \ldots,$$

we have

$$\|x_k - x^*\| \leq 2C_1 q^{2^k}, \quad C_1 > 0, \quad k = 0, 1, \ldots$$

Hence, the sequence $\{x_k\}$ converges to x^* quadratically.

Remark 2. *In the method (16) element h^* is given for better understanding of the idea this method, but for a practical implementation we have to construct also the element h_k on each step of the iteration taking into account some additional assumptions.*

The hereby mentioned procedure is described in [5].

Example 3. Consider the problem

$$x_1 - x_2 + x_1^2 \to \min$$

$$\text{subject to} \qquad F(x) = (x_1^2 + x_2^2)(x_2 - x_1) + x_1 x_2^3 = 0. \tag{17}$$

For this problem, we could not guarantee that the traditional methods converge to the solution. Moreover

$$F'(x,y) = \begin{pmatrix} 2x_1x_2 - 3x_1^2 - x_2^2 + x_2^3 \\ x_1^2 + 3x_2^2 - 2x_1x_2 + 3x_1x_2^2 \end{pmatrix}, \qquad F''(x,y) = \begin{pmatrix} 2x_2 - 6x_1 & 2x_1 - 2x_2 + 3x_2^2 \\ 2x_1 - 2x_2 + 3x_2^2 & 6x_2 - 2x_1 + 6x_1x_2 \end{pmatrix},$$

$$F'''(x)[h]^3 = -6(h_1^3 - h_1^2h_2 + h_1h_2^2 - h_2^3 - 3h_1h_2^2x_2 - h_2^3x_1) = 0.$$

Hence at the point x^*

$$\mathrm{Ker}^3 F'''(x^*) = \{\left(\begin{smallmatrix} t \\ t \end{smallmatrix}\right) \text{ and } t \neq 0\}.$$

We can assume that

$$h = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From the solution of the following system of equations

$$R(z) \;=\; R(x,\lambda) = \begin{pmatrix} \phi'(x) + \lambda(F'(x) + P_2F'''(x)[h]^2) \\ F(x) + P_2F''(x)[h]^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

namely from

$$R(z) \;=\; \begin{pmatrix} 2x_1+1+\lambda(2x_1x_2-3x_1^2-x_2^2+x_2^3+6x_2-4) \\ -1+\lambda(x_1^2+3x_2^2-2x_1x_2+3x_1x_2^2+12x_2+6x_1+4) \\ (x_1^2+x_2^2)(x_2-x_1)+x_1x_2^3+6x_2^2+6x_1x_2-4x_1+4x_2 \end{pmatrix} = \\ = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

we obtain a locally unique point $x^* = (0,0)$, $\lambda^* = \frac{1}{4}$.

Then

$$R'(z) = \\ \begin{pmatrix} 2+2\lambda x_2-6\lambda x_1 & \lambda(2x_1-2x_2+3x_2^2+6) \\ \lambda(2x_1-2x_2+3x_2^2+6) & \lambda(6x_2-2x_1+6x_1x_2+12) \\ 2x_1x_2-3x_1^2-x_2^2+x_2^3+6x_2-4 & x_1^2+3x_2^2-2x_1x_2+3x_1x_2^2+12x_2+6x_1+4 \end{pmatrix.\dots} \\ \begin{matrix} .... & 2x_1x_2-3x_1^2-x_2^2+x_2^3+6x_2-4 \\ ... & x_1^2+3x_2^2-2x_1x_2+3x_1x_2^2+12x_2+6x_1+4 \\ ... & 0 \end{matrix}\Bigg).$$

Now, substituting z^* into $R'(z)$ we infer that

$$R'(z^*) = \begin{pmatrix} 2 & 1,5 & -4 \\ 1,5 & 3 & 4 \\ -4 & 4 & 0 \end{pmatrix}$$

is nonsingular. Applying Theorem 4, we conclude that z^* is a regular local solution of the considered problem.

5. General case

In order to simplify the idea of the *p-factor Lagrange method* in the scheme (5) we have defined the element h at the point x^*. But now for practical implementation, we have to construct the element h_k on each iteration using some assumptions. In this part of our paper, we present the method in which we can find an approximation for h_k as well.

Let $h^* = P_{H_p(x^*)}\overline{h}$, and let $\operatorname{dist}(\overline{h}, H_p(x^*)) \le \varepsilon$ hold. For defining h^* let us consider auxiliary problem in the following form

$$\min_{h\in\mathbb{R}^n} \|\overline{h} - h\|^2 \tag{18}$$

subject to

$$F'(x)h + P_1F''(x)[h]^2 + \ldots + P_{p-1}F^p(x)[h]^p = 0.$$

We denote the solution to this problem by $h(x) := P_{H_p(x)}\overline{h}$.

Vector $\nu^* \in \mathbb{R}^m$ can be obtained from the equality

$$(\overline{h} - h^*) - (F'(x^*) + \ldots + P_{p-1}F^p(x^*)[h^*]^{p-1})^T \nu^* = 0. \tag{19}$$

We obtain this equality from Lagrange theorem applied to the problem (18) with variable h.

Introduce the notations

$$z := (x, \lambda, \nu, h)^T \in \mathbb{R}^{n+2m}, \quad z^* := (x^*, \lambda^*, \nu^*, h^*)^T,$$

and define the mapping $\Gamma : R^{n+2m} \to \mathbb{R}^{n+2m}$, as follows

$\Gamma(z) = \Gamma(x, \lambda, \nu, h) =$

$$\begin{pmatrix} \phi'(x) + (F'(x) + P_1 F''(x)[h] + \cdots + P_{p-1}F^{(p)}(x)[h]^{p-1})^T \lambda \\ F(x) + P_1 F'(x)[h] + \cdots + P_{p-1}F^{(p-1)}(x)[h]^{p-1} \\ 2(\overline{h} - h) - (F'(x) + \cdots + P_{p-1}F^p(x)[h]^{p-1})^T \nu \\ F'(x)h + P_1 F''(x)[h]^2 + \cdots + P_{p-1}F^p(x)[h]^p \end{pmatrix}. \tag{20}$$

Consider the system of equations $\Gamma(z) = 0$.

In this way we obtain system of $n+2m$ equations with $n+2m$ unknowns. The first equation is the necessary optimality condition for degenerate problems, the second one we can treat as a mapping, which is regular at the point x^*, the third is a system of equations (19) and the last m equation constitues the system of constraints of problem (18). Consider the p-**factor Lagrange method**, in the following form

$$\begin{pmatrix} x_{k+1} \\ \lambda_{k+1} \\ \nu_{k+1} \\ h_{k+1} \end{pmatrix} = \begin{pmatrix} x_k \\ \lambda_k \\ \nu_k \\ h_k \end{pmatrix} - \{\Gamma'(x_k, \lambda_k, \nu_k, h_k)\}^{-1} \cdot \Gamma(x_k, \lambda_k, \nu_k, h_k) \tag{21}$$

which is equivalent to $z_{k+1} = z_k - \{\Gamma'(z_k)\}^{-1} \cdot \Gamma(z_k)$ for $k = 0, 1, ...$

We introduce the following assumption, which is important for the differential properties of the $h(x)$. We need this assumption to guarantee the existences and the uniqueness of the projection $\overline{h}$ on the set $H_p(x^*)$(in our method on the set $H_p(x_k)$), which is important for the presented algorithm.

Assumption 1: The mapping F satisfies the normal condition of p-regularity at z^* with respect to a given element $\overline{h} \in \mathbb{R}^n$, $\|\overline{h}\| = 1$. Moreover, for some sufficiently small $\varepsilon > 0$, the element $\overline{h}$ satisfies

$$\text{dist}(\overline{h}, H_p(x^*)) \leq \varepsilon. \tag{22}$$

Lemma 2. *Suppose that Assumption 1 holds. Then there a exists sufficiently small $\varepsilon > 0$ such that the mapping $h(x)$ (defined above) is unique and continuously differentiable in a neighborhood $U_\varepsilon(x^*)$ of the point x^*.* (This lemma is proved in [5].)

Under the assumption 1 of Lemma 2 we have:

Theorem 5. *Let $x^* \in \mathbb{R}^n$ be a local minimizer to (1)–(2), $\phi \in \mathcal{C}^2(\mathbb{R}^n)$, $F \in \mathcal{C}^{p+1}(\mathbb{R}^m)$. Assume that F satisfies the normal condition of p-regularity at the point z^* with respect to $\overline{h}$ and let Assumption 1 hold, and assume that the matrix $\Gamma'(z)$ is nonsingular at z^*.*
Then, for sufficiently small $\varepsilon > 0$, the point z^* is a regular local isolated solution of the system $\Gamma(z) = 0$. Moreover, the sequence $\{z_k\}$ defined by method (21) converges to z^* with quadratic rate, i.e. $\|z_{k+1} - z^*\| \le C\|z_k - z^*\|^2$, for $k = 0, 1, \ldots$ where $C > 0$ is an independent constant.

Corollary 2. *Let $(x_0, y_0) \in U_\delta(x^*, y^*)$, $\lambda_0 \in U_\delta(\lambda^*)$, $h_0 \in U_\varepsilon(h^*)$ and $v_0 \in U_\delta(v^*)$, where $\delta > 0$ is sufficiently small. If the matrix $\Gamma'(z)$ is nonsingular at the point z^* then the sequence $\{x_k\}$, $k = 1, 2, \ldots$, converges to x^* quadratically, i.e. $\|x_k - x^*\| \le Cq^{2^k}$, $q < 1$ and $C > 0$, $k = 0, 1, \ldots$*

Example 4. Consider again the problem (17). But now let us use generalized method (21) for its solution. We construct the new system of equations in the following form
$\Gamma(z) = \Gamma(x, \lambda, \widehat{\lambda}, h) =$

$$\begin{pmatrix} \phi'(x) + (F'(x) + P_1 F''(x)[h] + \ldots + P_{p-1}F^{(p)}(x)[h]^{p-1})^T \lambda \\ F(x) + P_1 F'(x)[h] + \ldots + P_{p-1}F^{(p-1)}(x)[h]^{p-1} \\ 2(\overline{h} - h) - (\widehat{F}'(x) + \ldots + P_{p-1}F^p(x)[h]^{p-1})^T \widehat{\lambda} \\ \widehat{F}'(x)h \\ P_1 F''(x)[h]^2 \\ \ldots \\ P_{p-1}F^p(x)[h]^p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In this case, $P_1 = 0$, $P_2 \neq 0$ and the previous system is reduced to the system $\Gamma(z) = \Gamma(x, \lambda, \widehat{\lambda}, h) =$

$$\begin{pmatrix} \phi'(x) + (F'(x) + P_2 F'''(x)[h]^2)^T \lambda \\ F(x) + P_2 F''(x)[h]^2 \\ 2(\overline{h} - h) - (\widehat{F}'(x) + P_2 F'''(x)[h]^2)^T \widehat{\lambda} \\ P_2 F'''(x)[h]^3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Then the system of equations $\Gamma(z) = 0$ becomes

$$\begin{pmatrix} 2x_1+1+\lambda(2x_1x_2-3x_1^2-x_2^2+x_2^3-6h_1^2+4h_1h_2-2h_2^2+6h_2^2x_2) \\ -1+\lambda(x_1^2+3x_2^2-2x_1x_2+3x_1x_2^2+2h_1^2-4h_1h_2+6h_2^2+12x_2h_1h_2+6x_1h_2^2) \\ (x_1^2+x_2^2)(x_2-x_1)+x_1x_2^3+(2x_2-6x_1)h_1^2+(4x_1-4x_2+3x_2^2)h_1h_2+(6x_2-2x_1+6x_1x_2)h_2^2 \\ 2\overline{h}_1-2h_1-\widehat{\lambda}(-6h_1^2+4h_1h_2-2h_2^2+6h_2^2x_2) \\ 2\overline{h}_2-2h_2-\widehat{\lambda}(2h_1^2-4h_1h_2+6h_2^2+6h_2^2x_1+12h_1h_2x_2) \\ -6h_1^3+6h_1^2h_2-6h_1h_2^2+6h_2^3+18h_1h_2^2x_2+6h_2^3x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, $z^* = \begin{pmatrix} x^* \\ \lambda^* \\ \widehat{\lambda}^* \\ h^* \end{pmatrix}$ is the solution of above-mentioned system, where $x^* = \binom{0}{0}$, $\lambda^* = \frac{1}{4}$, $\widehat{\lambda}^* = \frac{1}{10}$ and $h^* = \binom{1}{1}$.

Now, substituting z^* into $\Gamma'(z^*)$ we get the matrix

$$\Gamma'(z^*) = \begin{pmatrix} 2 & 1.5 & -4 & 0 & -2 & 0 \\ 1.5 & 3 & 4 & 0 & 0 & 2 \\ -4 & 4 & 0 & 0 & 0 & 0 \\ 0 & -0.6 & 0 & 4 & -1.2 & 0 \\ -0.6 & 12 & 0 & -4 & 0 & -2.8 \\ 6 & 18 & 0 & 0 & -12 & 12 \end{pmatrix}$$

which is nonsingular. Applying Theorem 5, we conclude that z^* is a regular local solution of (17).

Numerical results:

Applying the described method to the same problem (17) with the initial point

$$\begin{cases} x_0 = \begin{pmatrix} 0{,}5 \\ 0{,}5 \end{pmatrix} \\ \lambda_0 = 0{,}15 \\ \widehat{\lambda}_0 = 0{,}4 \\ h_0 = \begin{pmatrix} 0{,}6 \\ 0{,}7 \end{pmatrix} \end{cases}$$

we obtain in 4-th iteration a good approximation of the solution point $x^* = \binom{0}{0}$, $\lambda^* = \frac{1}{4}$, $\widehat{\lambda}^* = \frac{1}{10}$ and $h^* = \binom{1}{1}$.

$$\text{k=1} \begin{cases} x_1 = \begin{pmatrix} -0{,}0135 \\ 0{,}3356 \end{pmatrix} \\ \lambda_1 = 0{,}3674 \\ \widehat{\lambda}_1 = -0{,}2157 \\ h_1 = \begin{pmatrix} 1{,}1851 \\ 1{,}0456 \end{pmatrix} \end{cases} \quad \text{k=2} \begin{cases} x_2 = \begin{pmatrix} 0{,}0152 \\ 0{,}0232 \end{pmatrix} \\ \lambda_2 = 0{,}2559 \\ \widehat{\lambda}_2 = 0{,}0649 \\ h_2 = \begin{pmatrix} 1{,}0893 \\ 1{,}0807 \end{pmatrix} \end{cases} \quad \text{k=3} \begin{cases} x_3 = \begin{pmatrix} 0{,}0009 \\ 0{,}0001 \end{pmatrix} \\ \lambda_3 = 0{,}2524 \\ \widehat{\lambda}_3 = 0{,}0953 \\ h_3 = \begin{pmatrix} 1{,}0013 \\ 1{,}0000 \end{pmatrix} \end{cases} \quad \text{k=4} \begin{cases} x_4 = \begin{pmatrix} 0{,}0000 \\ 0{,}0000 \end{pmatrix} \\ \lambda_4 = 0{,}2500 \\ \widehat{\lambda}_4 = 0{,}1000 \\ h_4 = \begin{pmatrix} 1{,}0000 \\ 1{,}0000 \end{pmatrix} \end{cases}$$

For the problem considered above we could not use the classical methods for example Newton‘s method because the first derivatives is degenerate $F'(x^*) = 0$.

6. Identification of linear independent constraints algorithm

In this paper we consider methods for solving degenerate optimization problems in which we use the same orthoprojectors like $P_1, P_2, ..., P_m$. But in practical implementation it is hard to construct this type of oprtoprojections, so in order to avoid difficult calculations we construct new mapping $\widehat{F}(x)$ equivalent to the mapping $F(x)$. We assume for this mapping that the first r rows in the Jacobian matrix $\widehat{F}\,'(x^*)$ are linearly independent, and the others are equal to zero,

$$\widehat{F}'(x^*) = \begin{pmatrix} f_1'(x^*) \\ \vdots \\ f_r'(x^*) \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{23}$$

Any system of nonlinear equations can be transformed into a new one for which (23) holds. This transformation is performed by multiplication of the original mapping $F(x)$ by the nondegenerate matrix, it means $F(x) = 0 \Longleftrightarrow \widehat{F}(x) = 0$ (see [10]). So under this transformation, the feasible set $\{x \mid F(x) = 0\}$ does not change.

Throughout this section we consider the algorithm in which we identify the constraints that are linearly independent at the solution point x^* using information in any given point $x_0 \in U(x^*)$.

Let the mapping $F \in \mathcal{C}^{p+1}(\mathbb{R}^n)$ be p- regular ($p \geq 2$) at the point x^*. For sufficiently small neighborhood $U(x^*)$ we introduce two functions

$$\rho(x) = \min_{i=1,\cdots,m} \operatorname{dist}\left(f_i'(x), \operatorname{Span}\left(f_j'(x),\, j \in \{1,...,m\} \setminus i\right)\right),$$

$$\mu(x) = \max\left\{\|F(x)\|^{\frac{1}{p}}, \rho(x)^{\frac{1}{p-1}}\right\}.$$

Here f_i is the ith coordinate of vector function F and $\operatorname{Span}(a_1, \ldots, a_m)$ is a linear span of vectors $a_1, \ldots, a_m$.

The following theorem describes the function $\mu(x)$.

Theorem 6. *Let the mapping $F \in \mathcal{C}^{p+1}(\mathbb{R}^n)$ be p-regular at the point x^* and $F(x^*) = 0$.*
Then there exist $\varepsilon > 0$, $C' > 0$, $C'' > 0$, such that following inequality

$$C'\|x - x^*\| \leq \mu(x) \leq C''\|x - x^*\|^{\frac{1}{p}}, x \in U_\varepsilon(x^*)$$

holds.

In order to obtain mappings as follows

$$\widehat{F}(x) = \begin{pmatrix} f_1(x) \\ \ldots \\ f_{r_1}(x) \\ \widehat{f}_{r_1+1}(x) \\ \ldots \\ \widehat{f}_m(x) \end{pmatrix} \tag{24}$$

we consider the method in which we identify the gradients $f_1'(x), f_2'(x), \ldots, f_{r_1}'(x)$ that are linearly dependent at the solution point x^*.

To construct the method, we need the following lemma.

Lemma 3. *Let for nonnegative functions $g(x), \mu(x)$ the following inequality be satisfied:*

$$|g_1(x) - g_2(x)| \leq L\|x_1 - x_2\|,\ \forall x_1, x_2 \in U_\delta(x^*),$$

$$C'\|x - x^*\| \leq \mu(x) \leq C''\|x - x^*\|^{\frac{1}{p}}, x \in U_\delta(x^*),$$

where L, δ, C', C'', are positive constants and $C'' \geq C'$, $p \geq 2$.

Then there exists a sufficiently small $\varepsilon > 0$, such that one of the following conditions holds:
1. $\forall x \in U_\varepsilon(x^*):\ g(x) \leq \mu(x)^{\frac{1}{2}}$ *and then* $g(x^*) = 0$
or
2. $\forall x \in U_\varepsilon(x^*):\ g(x) > \mu(x)^{\frac{1}{2}}$ *and then* $g(x^*) \neq 0$.

Remark 3. *Under the assumptions of Lemma 3 there exists sufficiently small $\varepsilon > 0$ such that if for some element $\overline{x} \in U_\varepsilon(x^*)$ the following inequality $g(\overline{x}) \le \mu(\overline{x})^{\frac{1}{2}}$ holds then for all $x \in U_\varepsilon(x^*)$ we have $g(x) \le \mu(x)^{\frac{1}{2}}$, so equation $g(x^*) = 0$ is fulfilled. Analogously, if for some element $\overline{x} \in U_\varepsilon(x^*)$ the following inequality $g(\overline{x}) > \mu(\overline{x})^{\frac{1}{2}}$ holds then for all $x \in U_\varepsilon(x^*)$ we have $g(x) > \mu(x)^{\frac{1}{2}}$, so condition $g(x^*) \ne 0$ is fulfilled.*

Method for identification the constraints that are linearly independent at the point x^*

For sufficiently small $\varepsilon > 0$, $x \in U_\varepsilon(x^*)$, $i \in \{1, ..., m\}$ and under the assumptions of Lemma 3 and Remark 3 we consider two cases:

$$\begin{array}{ll} 1. & g(x) = \|f_i'(x)\| \le \mu(x)^{\frac{1}{2}} \\ 2. & g(x) = \|f_i'(x)\| > \mu(x)^{\frac{1}{2}}. \end{array} \tag{25}$$

In case 1 under Remark 3 we have $f_i'(x^*) = 0$ and analogously in case 2 we have that $f_i'(x^*) \ne 0$.

Algorithm

Step 1. Applying the scheme (25) we identify the first vector which is nonzero. At the beginning let this vector have index i_1 and let the inequality $\|f_{i_1}'(x)\| > \mu(x)^{\frac{1}{2}}$ be fulfilled then the corresponding vector is nonzero $f_{i_1}'(x^*) \ne 0$.

Step 2. We identify the next vector which is nonzero for example $f_{i_2}'(x^*)$ and check the condition $\text{dist}\left(f_{i_2}'(x), \text{Span}\left(f_{i_1}'(x)\right)\right) > \mu(x)^{\frac{1}{2}}$. If this condition holds in $U_\varepsilon(x^*)$ then, it means that the vectors$f_{i_1}'(x^*), f_{i_2}'(x^*)$ are linearly independent and we go to **step** 3. But if this condition does not hold, it means the vector $f_{i_2}'(x^*)$ is a linear combination of the other vectors and we go again to the **step** 2.

Step 3. We identify the next vector which is nonzero for example $f_{i_3}'(x^*)$ and check the condition $\text{dist}\left(f_{i_3}'(x), \text{Span}\left(f_{i_1}'(x), f_{i_2}'(x)\right)\right) > \mu(x)^{\frac{1}{2}}$. If this condition holds in $U_\varepsilon(x^*)$ then, it means that the vector $f_{i_3}'(x^*)$ is linearly independent of the other vectors and we go to the next step. But if this condition does not hold, it means the vector $f_{i_3}'(x^*)$ is a linear combination of the other vectors and we go again to the **step** 3.

Step s. We have $f_{i_1}'(x^*), f_{i_2}'(x^*), ..., f_{i_{s-1}}'(x^*)$ linearly independent vectors. Now we again identify the vector which is nonzero for example $f_{i_s}'(x^*)$ and check the condition $\text{dist}\left(f_{i_3}'(x), \text{Span}\left(f_{i_1}'(x), f_{i_2}'(x), ..., f_{i_s}'(x)\right)\right) > \mu(x)^{\frac{1}{2}}$. If this condition holds in $U_\varepsilon(x^*)$ then, it means that the vector $f_{i_s}'(x^*)$ is linearly independent of the other vectors. But if this conditions does not hold, it means the vector $f_{i_s}'(x^*)$ is a linear combination of the other vectors.

This algorithm stops when $s = m$.

In this way we obtain the mapping of the form (24) where $f_{i_1}'(x^*) \ne 0, ..., f_{i_{r_1}}'(x^*) \ne 0$, are linearly independent but $f_{i_{r_1+1}}'(x^*) = 0, ..., f_{i_m}'(x^*) = 0$ are dependent. Analogously we procede with second derivatives.

Transformation of the original mapping $F(x)$ to the mapping $\widehat{F}(x)$ has been carried in order to simplify oprhoprojectors. But in the hereby paper we assumed that mapping $F(x)$ fulfills (23).

Under this transformation for the mapping $\widehat{F}(x)$ in the case $p = 2$ we have the following orthoprojections $P_1 = \begin{pmatrix} 1 & & & & \\ & \cdots & & & \\ & & 1_{r_1} & & \\ & & & 0 & \\ & & & & \cdots \\ & & & & & 0 \end{pmatrix}$ and $P_2 = \begin{pmatrix} 0 & & & & \\ & \cdots & & & \\ & & 0 & & \\ & & & 1_{r_1+1} & \\ & & & & \cdots \\ & & & & & 1_m \end{pmatrix}$, which have simple structure.

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